

# Equivariant Solutions to a System of Nonlinear Wave Equations with Ginzburg-Landau Type Potential

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## Abstract

It is known that there exist solutions with interfaces to various scalar nonlinear wave equations. In this paper, we look for solutions of a two-component system of nonlinear wave equations where one of the components has an interface and where the second component is exponentially small except near the interface of the first component. A formal asymptotic expansion suggests that there exist solutions to this system with these characteristics whose profiles are determined by the winding number density of the second component and where the interface of the first component is a time-like surface in Minkowski space whose geometric evolution is coupled in a highly nonlinear way to the phase of the second component. We verify this heuristic when  $n = 2$  and for equivariant maps.

## 1 Introduction

### 1.1 Synopsis

In this paper we consider two-component systems of hyperbolic system of PDEs qualitatively similar to

$$\begin{cases} \partial_{tt}\phi - \Delta\phi + \frac{\lambda_\phi}{\epsilon^2}(\phi^2 - 1)\phi = -\frac{\beta}{\epsilon^2}|\sigma|^2\phi \\ \partial_{tt}\sigma - \Delta\sigma + \frac{\lambda_\sigma}{\epsilon^2}(|\sigma|^2 - 1)\sigma = -\frac{\beta}{\epsilon^2}\phi^2\sigma \end{cases} \quad (1.1)$$

where  $\Phi := (\phi, \sigma) : \mathbb{R}^{1+n} \rightarrow \mathbb{R} \times \mathbb{C}$ ,  $0 < \epsilon \ll 1$  is a small parameter of the model, and  $(\lambda_\phi, \lambda_\sigma, \beta)$  are real, non-negative constants. We are interested in solutions to (1.1) with the properties that

- $\phi$  has an interface
- $\sigma$  is exponentially small except near the interface

For the first equation of (1.1), if the right hand side vanishes (which happens if  $\beta = 0$  or if  $\sigma = 0$ ), then it is known that there exists a  $\phi$  with an interface solving this equation [14]. We, however, are interested in regimes where  $\phi$  and  $\sigma$  are coupled (i.e.  $\beta \neq 0$ ) and where  $(\lambda_\phi, \lambda_\sigma, \beta)$  are chosen so that  $(\phi, \sigma)$  have the properties described above, which in particular stipulate that  $\sigma \neq 0$  near the interface of  $\phi$ . For these regimes, it follows from the physics literature on superconducting strings, reviewed in section 2 below, that the  $\sigma$ -field can naturally be identified with a superconducting current confined to the interface of  $\phi$ . Hence, we call (1.1) the **superconducting interface model**. The goal of this paper is to understand the coupling between the current and the interface and, in particular, how the current affects the dynamics of the interface.

As discussed in appendix A, a formal asymptotic expansion suggests that for suitable local coordinates  $(y^\tau, y^\nu) = (y_0, \dots, y_n)$  near a codimension one time-like surface  $\Gamma$ , with  $y^\tau = (y_0, \dots, y_{n-1})$  parameterizing  $\Gamma$  and with  $\{y^\nu = 0\}$  corresponding to  $\Gamma$ , then there *should* exist a solution to (1.1) satisfying

$$\begin{cases} \phi(y^\tau, y^\nu) \approx \phi_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \\ \sigma(y^\tau, y^\nu) \approx e^{\frac{i}{\epsilon}\theta(y^\tau)} \sigma_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \end{cases} \quad (1.2)$$

where

(1a)  $\theta$  is a function of  $y^\tau$  only

(1b)  $\zeta(y^\tau) := \gamma(\nabla_\tau \theta, \nabla_\tau \theta)$ , where  $\nabla_\tau$  denotes the tangential gradient along  $\Gamma$  and  $\gamma_{ij}$  is the induced metric on  $\Gamma$  (the ambient metric for this problem is the Minkowski metric - denoted  $\eta$ ).

(1c) For each  $\rho \in \mathbb{R}$  we have that  $\Phi_0(\cdot; \rho) := (\phi_0, \sigma_0)(\cdot; \rho) : \mathbb{R} \rightarrow \mathbb{R}^2$  satisfies the minimization problem

$$\begin{aligned} \mu(\rho) &= \inf_{(f,s) \in \mathcal{A}} \int \left\{ \frac{1}{2} |(f', s')|^2 + V(f, s) + \frac{1}{2} \rho s^2 \right\} \\ \mathcal{A} &:= \left\{ (f, s) \in C^1(\mathbb{R}, \mathbb{R}^2) : \lim_{y^\nu \rightarrow \pm\infty} f(y^\nu) = \pm 1 \right\} \end{aligned}$$

for suitable potentials  $V$ . In particular, the the profiles  $\phi_0$  and  $\sigma_0$  in (1.2) are determined by  $\zeta(y^\tau)$ .

(1d)  $\theta$  and  $\Gamma$  satisfy the highly nonlinear, coupled system of PDEs

$$\square_\Gamma \theta = -\gamma(\nabla_\tau \log[\mu'(\zeta)], \nabla_\tau \theta) \quad (1.3)$$

$$\text{Mean Curvature of } \Gamma = 2 \frac{\mu'(\zeta)}{\mu(\zeta)} \mathbb{I}(\nabla_\tau \theta, \nabla_\tau \theta) \quad (1.4)$$

where the ambient metric that the mean curvature and the second fundamental form  $\mathbb{I}$  are defined with respect to is the Minkowski metric.

In this paper, we verify, subject to a non-degeneracy condition, that there does indeed exist a solution to (1.1) satisfying (1.2) when  $n = 2$  and when  $\Phi$  is an equivariant map.

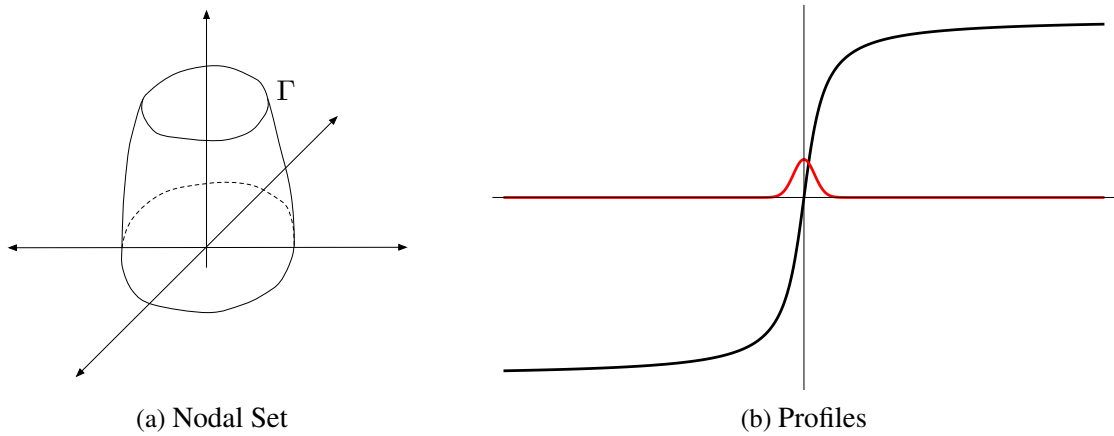


Figure 1: The formal asymptotic expansion suggests that there exists a solution  $\Phi = (\phi, \sigma)$  to (1.1) so that for  $\theta$  and  $\Gamma$  satisfying (1.3 - 1.4), then at each  $p \in \Gamma$  we expect that as we move away from  $\Gamma$  in the transverse direction  $\phi$  looks like the black curve in (1b) and  $\sigma$  looks like  $e^{\frac{i}{\epsilon}\theta(s_M(p))} \sigma_0$  where  $\sigma_0$  looks like the red curve in (1b). Looking at figure 1a, this means that  $\sigma$  is exponentially small except near  $\Gamma$ ,  $\phi \approx -1$  inside  $\Gamma$ ,  $\phi \approx 1$  outside of  $\Gamma$ , and  $\phi$  transitions from  $-1$  to  $1$  near  $\Gamma$ .

It can be shown that if the winding number density  $\gamma^{ij}\partial_i\theta\partial_j\theta$  is sufficiently large, then the  $\sigma_0$ -field of the approximate solution is 0. It is believed that there are regimes where a solution may initially have a non-zero current (i.e.  $\sigma_0(\frac{1}{\epsilon}\gamma^{ij}\partial_i\theta\partial_j\theta) \neq 0$ ), but as the system evolves the solution may lose its current. This type of phenomena is referred to as **current quenching** [25] and we show in section 3.4.2 below that given suitable initial conditions that the solutions we find undergo current quenching.

To the best of our knowledge this is the first paper to consider interface type solutions to a two-component, hyperbolic system.

### 1.1.1 Mathematical Background

There is an extensive mathematical literature with results that are of the type we show in this paper. The unifying theme of these types of results is

- There exist solutions to some PDEs which have interfaces, point vortices, or vortex filaments whose dynamics are approximately described by some associated geometric problem.

See [14] for a detailed account of these types of results for the scalar elliptic, parabolic, and hyperbolic counterparts of (1.1).

Two-component systems have been considered in the physics literature as models for interfaces, point vortices, or vortex filaments in various physical systems [16, 12]. However, rigorous mathematical descriptions of solutions to two-component systems of the type we consider are sparse in the math literature. For example, progress on the existence and classification of solutions with interfaces or vortices has been made for various two-component, elliptic systems [3, 5, 6, 4] and (potentially very complicated) ground states of other two-component models subject to physically relevant forcing has been studied [18, 1, 2].

A scalar analogue of (1.1) is

$$\partial_{tt}u - \Delta u + \frac{1}{\epsilon^2}V'(u) = 0 \quad (1.5)$$

where  $u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  and  $V'(u)$  is qualitatively similar to

$$\lambda(|u|^2 - 1)u$$

In this case, it has been shown that there exists a solutions to (1.5) that have an interface near a codimension one time-like minimal surface [14, 10]. These results are obtained using weighted energy estimates to show that if one starts with appropriate initial data, then there exists an exact solution to (1.5) that is close to an approximate solution obtained using formal arguments.

One can also consider a version of (1.5) for which  $u : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$ . In this case, the goal is to find and describe solutions to (1.5) with vortices or vortex filaments. Results describing point vortices and/or vortex filaments in (1.5) and a gauged version of (1.5) have been obtained in [13, 17, 14] and [11, 9], respectively. Similarly for us, we could consider the case when  $\phi : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$ . In this case, we would like to find solutions to (1.1) so that  $\phi$  has a vortex filament and  $\sigma$  is exponentially small except near the vortex filament of  $\phi$ , but for now we focus our attention on the case where  $\phi : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  and has an interface.

In contrast to [14, 10] who use weighted energy estimates, as in [22, 23, 11] we linearize (1.1) about an approximate solution obtained using a formal asymptotic expansion and we use spectral properties of the linearized operator to show that there exists an exact solution of (1.1) which is close to the approximate solution. The reason we use a different approach is that in order to resolve the new complexities introduced by the coupling of the current to the interface of  $\phi$ , a more detailed description of solutions is required that seems hard to obtain using weighted energy estimates.

## 1.2 Description of Results

We will simplify (1.1) by considering the case when  $\Phi = (\phi, \sigma) : \mathbb{R}^{1+2} \rightarrow \mathbb{R} \times \mathbb{C}$  is equivariant. That is, we assume  $\phi$  and  $\sigma$  are of the form

$$\begin{aligned}\phi(t, x) &= \tilde{\phi}(t, |x|) \\ \sigma(t, x) &= e^{i \frac{d}{\epsilon} \arg(x)} \tilde{\sigma}(t, |x|)\end{aligned}\tag{1.6}$$

for  $(\tilde{\phi}, \tilde{\sigma}) : \mathbb{R}^{1+1} \rightarrow \mathbb{R}^2$  and  $d \in \mathbb{R}/2\pi\epsilon\mathbb{Z}$  is a fixed constant. Using (1.6) to simplify, then for  $(t, r) \in \mathbb{R} \times \mathbb{R}_+$  we have that (1.1) becomes

$$\partial_{tt} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \partial_{rr} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \frac{1}{r} \partial_r \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} + \frac{1}{\epsilon^2} \begin{pmatrix} \lambda_\phi(\tilde{\phi}^2 - 1)\tilde{\phi} + \beta\tilde{\sigma}^2\tilde{\phi} \\ \lambda_\sigma(\tilde{\sigma}^2 - 1)\tilde{\sigma} + \beta\tilde{\phi}^2\tilde{\sigma} \end{pmatrix} + \frac{1}{\epsilon^2} \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{r^2} \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We will actually consider the more general family of equations

$$\partial_{tt} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \partial_{rr} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \frac{1}{r} \partial_r \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} + \frac{1}{\epsilon^2} \nabla_\Phi V(\tilde{\phi}, \tilde{\sigma}) + \frac{1}{\epsilon^2} \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{r^2} \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\tag{1.7}$$

where  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\nabla_\Phi V(\phi, \sigma) := (\partial_\phi V, \partial_\sigma V)$ . The initial data of (1.7) we consider is described in section 1.2.1 below. We will be interested in solutions to (1.7) for the rest of the paper. Hence, we will drop the  $\sim$ 's from  $\tilde{\phi}$  and  $\tilde{\sigma}$  for notational convenience.

Consider potentials  $V$  satisfying the following assumptions

1.  $V \in C^2(\mathbb{R}^2, \mathbb{R})$  and  $V(\phi, \sigma) = V(|\phi|, |\sigma|)$
2.  $V(\pm 1, 0) = 0$  and  $V(x, y) > 0$  for all  $(x, y) \neq (\pm 1, 0)$ . Furthermore,  $V(1, y) < V(x, y)$  for  $x > 1$ ,  $V(x, 1) < V(x, y)$  for  $y > 1$ , and  $V$  has a local maximum at  $(0, 0)$  with  $\nabla_\Phi V \neq 0$  on  $(-1, 1) \times (-1, 1)$  otherwise.
3.  $|\text{Hess}_\Phi V(\Phi)| \lesssim 1 + |\Phi|^2$  and  $\text{Hess}_\Phi V(\pm 1, 0) \geq \lambda_* I$  where  $I$  is the  $2 \times 2$  identity matrix and  $\lambda_* > 0$ .
4.  $V$  satisfies a non-degeneracy and a continuity condition - see (1.21) below.

For potentials  $V$  satisfying (1.8), we will construct solutions to (1.7) so that the  $\phi$ -field has an interface near a codimension one time-like surface satisfying some geometric problem.

Define

$$W(\Phi, R) := V(\Phi) + \frac{1}{2} \frac{d^2}{R^2} \sigma^2\tag{1.9}$$

and we call  $W$  the **shifted potential**. We will denote the gradient of the shifted potential as

$$w(\Phi, r) := \nabla_\Phi W(\Phi, r)\tag{1.10}$$

Let  $\Gamma$  be a codimension one surface parameterized by  $(\tau, R(\tau))$  representing the interface of  $\phi$  to be determined by (1.7).

**Lemma 1.2.1.** *Let*

$$\Gamma_T = \{(\tau, R(\tau)) : 0 \leq \tau \leq T \text{ where } T \text{ is the time of existence of } R \text{ with } |R'| < 1\}$$

*Then there exists a neighbourhood  $\mathcal{N}$  (independent of  $\epsilon$ ) of  $\Gamma_T$  on which there exists a differentiable solution to*

$$\begin{cases} -\partial_t d_M^2 + \partial_r d_M^2 = 1 & \text{on } \mathcal{N} \\ d_M = 0 & \text{on } \Gamma_T \end{cases} \quad (1.11)$$

*Furthermore, there exists  $s_M : \mathbb{R}^{1+1} \rightarrow \mathbb{R}$  satisfying*

$$\begin{aligned} -\partial_t d_M \partial_t s_M + \partial_r d_M \partial_r s_M &= 0 & \text{on } \mathcal{N} \\ (s_M(t, r), R(s_M(t, r))) &= (t, r) & \text{on } \Gamma_T \end{aligned} \quad (1.12)$$

*so that*

$$(t, r) = (s_M(t, r), R(s_M(t, r))) + \frac{d_M(t, r)}{\sqrt{1 - R'(s_M(t, r))^2}} (R'(s_M(t, r)), 1) \quad (1.13)$$

A proof for lemma 1.2.1 can be found in [15].

Since  $d_M$  is a continuous, then there exists  $c > 0$  so that

$$\Sigma_{c,T} := \{(t, r) : \text{for } 0 \leq t \leq T \text{ and } d_M(t, r) \leq c\} \subset \mathcal{N} \quad (1.14)$$

where we possibly take  $T$  smaller. The initial data of (1.7), specified in section 1.2.1, is chosen so that  $\Phi$  transitions from  $(-1, 0)$  to  $(1, 0)$  on  $\Sigma_{c,T}$  and so that  $\Phi$  is either  $(-1, 0)$  or  $(1, 0)$  outside of  $\Sigma_{c,T}$ .

We look for solutions of (1.7) of the form

$$\Phi(t, r) \approx F_0\left(\frac{d_M}{\epsilon}; R(s_M)\right) + \epsilon F_1\left(\frac{d_M}{\epsilon}; R(s_M), R'(s_M)\right) \quad (1.15)$$

We could use the same notation as we use in appendix A and write  $F_0 = F_0(x; \frac{d^2}{R^2})$ , but we write  $F_0 = F_0(x; R)$  for convenience. As for the  $F_1$  term, we tried to show that there exists a solution to (1.7) so that

$$\Phi \approx F_0\left(\frac{d_M}{\epsilon}; R(s_M)\right)$$

but when  $\sigma \neq 0$  the coupling between the  $\phi$ -field and  $\sigma$ -field introduces new subtleties into the nature of the solutions that necessitates a more detailed description. Hence, we consider the leading order correction  $F_1$ . We will see momentarily why  $F_1$  depends additionally upon  $R'$ .

Plugging  $F_0 + \epsilon F_1$  and  $r = R(s_M(t, r)) + \frac{d_M(t, r)}{\sqrt{1 - R'(s_M(t, r))^2}}$  into (1.7) we find that

$$\frac{1}{\epsilon^2} \text{ term: } F_0''(\partial_t d_M^2 - \partial_r d_M^2) + \nabla_\Phi W(F_0, R) \quad (1.16)$$

$$\begin{aligned} \frac{1}{\epsilon} \text{ term: } & F_1''(\partial_t d_M^2 - \partial_r d_M^2) + \text{Hess}_\Phi W(F_0, R) F_1 + F_0'(\partial_t d_M - \partial_{rr} d_M - \frac{1}{r} \partial_r d_M) \\ & - \frac{2}{\sqrt{1 - R'}} \frac{d_M(t, r)}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{R^3} \end{pmatrix} F_0 + 2\partial_R F_0' R'(\partial_t d_M \partial_t s_M - \partial_r d_M \partial_r s_M) \end{aligned} \quad (1.17)$$

where  $F_0 = F_0(\frac{d_M}{\epsilon}; R(s_M))$ ,  $F_1 = F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M))$ ,  $F_i'$  is the derivative of  $F_i$  with respect to the first coordinate  $x$ , and  $\partial_R F_0$  is the derivative of  $F_0$  with respect to the second coordinate of  $F_0$ . There

are lower order terms, but these are the two dominate terms. Using the fact that  $-\partial_t d_M^2 + \partial_r d_M^2 = 1$ ,  $-\partial_t d_M \partial_t s_M + \partial_r d_M \partial_r s_M = 0$ , and the fact that  $H(R) := -\partial_{tt} d_M + \partial_{rr} d_M + \frac{1}{r} \partial_r d_M$  is the mean curvature of the surface of rotation generated by  $R$  in  $\mathbb{R}^{1+2}$ , then (1.16) and (1.17) can be re-written as

$$\begin{aligned} \frac{1}{\epsilon^2} \text{ term: } & -F_0'' + \nabla_\Phi W(F_0, R) \\ \frac{1}{\epsilon} \text{ term: } & -F_1'' + \text{Hess}_\Phi W(F_0, R) F_1 - H(R) F_0' - \frac{2}{\sqrt{1-R'}} \frac{d_M(t, r)}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{R^3} \end{pmatrix} F_0 \end{aligned}$$

Heuristically, we find that for  $R \in \mathbb{R}$  if  $F_0 = (f_0, s_0)(x; R)$  solves

$$\begin{aligned} -F_0'' + \nabla_\Phi W(F_0, R) &= 0 \\ \lim_{x \rightarrow \pm\infty} F_0(x; R) &= \pm 1 \end{aligned} \tag{1.18}$$

and for  $L_1(F_0; R)$  defined in (1.20) below if  $F_1 = (f_1, s_1)(x; R, R')$  solves

$$\begin{aligned} L_1(F_0(x; R), R) F_1 &= H(R) F_0'(x; R) + 2 \frac{x}{\sqrt{1-R'^2}} \frac{d^2}{R^3} \begin{pmatrix} 0 \\ s_0(x; R) \end{pmatrix} \\ \lim_{x \rightarrow \pm\infty} F_1(x; R, R') &= 0 \end{aligned} \tag{1.19}$$

then  $F_0(\frac{d_M}{\epsilon}; R(s_M)) + \epsilon F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M))$  has the properties that we are looking for and looks to be a good approximate solution. Note that  $F_0$  and  $F_1$  depend on this so far unknown function  $R$  parameterizing  $\Gamma$ . In fact,  $F_0$  depends on  $R$  and  $F_1$  depends on  $R, R'$ , and  $R''$ . We will see momentarily that  $F_1$  actually only depends on  $R$  and  $R'$ .

Differentiating (1.18) with respect to  $x$  yields

$$-(F_0')'' + \text{Hess}_\Phi W(F_0; R) F_0' = 0$$

Define

$$L_1 := -\frac{d^2}{dx^2} I_{2 \times 2} + \text{Hess}_\Phi W(F_0; R) \tag{1.20}$$

We assume that

$$\ker(L_1(F_0; R)) = \text{span}\{F_0'\} \tag{1.21}$$

holds for  $R \in (r_0, r_1)$  with  $0 \leq r_0 < r_1 \leq \infty$ . This is the non-degeneracy condition we assume  $V$  satisfies. Further, we also assume that for  $F_0(\cdot; R)$  satisfying (1.18),

$$R \mapsto F_0(\cdot; R) \tag{1.22}$$

is a continuous map for all  $R \in (r_0, r_1)$ . This is the continuity condition we assume  $V$  satisfies. Of particular interest to us are potentials  $V$  for which (1.21) holds for a range of  $R$  for which  $s_0(\cdot; R) \neq 0$ .

The reason we assume (1.21) and (1.22) is they allow us to conclude that  $F_0(\cdot; R)$  is actually  $C^2$  in  $R$ . They also give us certain spectral estimates (see theorem 3.2.1) which will be important in the proof of theorem 1.2.2. Further, we will need to use (1.21) to find  $F_1$  solving (1.19) and decaying at infinity in proposition 3.3.4. In fact, to show that such an  $F_1$  exists, we need to solve an equation like  $L_1(F_0; R)f = g$

with  $g \in L^2(\mathbb{R}; \mathbb{R}^2)$ . A necessary condition for this to be solvable is that  $g \in \ker(L_1(F_0; R))^\perp$  where  $\perp = \perp_{L^2}$ . This necessary condition plus (1.21) suggests to us that for (1.15) to hold, then  $R$  must solve

$$H(R) \int |F'_0(x; R)|^2 dx - \frac{1}{\sqrt{1-R'^2}} \frac{d^2}{R^3} \int |s_0(x; R)|^2 dx = 0 \quad (1.23)$$

$$R(0) \in (r_0, r_1) \text{ and } R'(0) = 0$$

Since  $R(s_M)$  could leave  $(r_0, r_1)$ , we have that the approximate solution  $F_0(\frac{d_M}{\epsilon}; R(s_M)) + \epsilon F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M))$  is only valid up to some finite time  $T$  as  $F_1$  is only guaranteed to exist for as long as  $R(s_M) \in (r_0, r_1)$ . Furthermore, the mean curvature  $H(R)$  of  $\Gamma$  contains a  $R''$  term. Using (1.23), one can express  $R''$  in terms of  $R$  and  $R'$ . From this we see that  $F_1$  actually only depends on  $R$  and  $R'$  as stated earlier.

The main result obtained in this paper shows that  $F_0(\frac{d_M}{\epsilon}; R) + \epsilon F_1(\frac{d_M}{\epsilon}; R, R')$  is indeed a good approximate solution.

**Theorem 1.2.2.** *Given suitable initial conditions (see section 1.2.1 below), there exists a solution  $\Phi$  to (1.7), a function  $a : \mathbb{R} \rightarrow \mathbb{R}$ , and constants  $\bar{T} \leq T$  and  $\delta > 0$  (both independent of  $\epsilon$ ) with*

$$[0, \bar{T}] \times [\mathbb{R}_+ \setminus (R(0) - \delta, R(0) + \delta)] \cup \Sigma_{c, \bar{T}} = [0, \bar{T}] \times \mathbb{R}_+$$

so that on  $\Sigma_{c, \bar{T}}$ ,  $R(s_M(t, r)) \in (r_0, r_1)$  and

$$\left\| \Phi - F_0\left(\frac{d_M - a(s_M)}{\epsilon}; R(s_M)\right) - \epsilon F_1\left(\frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M)\right) \right\|_{L_t^1 H_r^1(\Sigma_{c, \bar{T}})} \lesssim \epsilon^2 \quad (1.24)$$

$$\left\| \partial_t \left[ \Phi - F_0\left(\frac{d_M - a(s_M)}{\epsilon}; R(s_M)\right) - \epsilon F_1\left(\frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M)\right) \right] \right\|_{L_t^1 L_r^2(\Sigma_{c, \bar{T}})} \lesssim \epsilon^2 \quad (1.25)$$

where  $d_M$  and  $s_M$  are functions of  $(t, r)$  and off of  $\Sigma_{c, \bar{T}}$

$$\begin{aligned} \Phi &= (-1, 0) \quad \text{for } (t, r) \in [0, \bar{T}] \times [0, R(0) - \delta] \\ \Phi &= (1, 0) \quad \text{for } (t, r) \in [0, \bar{T}] \times (R(0) + \delta, \infty) \end{aligned}$$

Most importantly, this theorem tells us that there exists a solution  $\Phi$  to (1.7) with the properties that we want. Namely, there exists a solution  $\Phi = (\phi, \sigma)$  so that  $\phi$  has an interface and for appropriate potentials  $V$ ,  $\sigma$  is exponentially small except near the interface of  $\phi$ .

### 1.2.1 Initial Data and the Existence of Solutions

Provided suitable initial data, to be described shortly, showing that the superconducting interface model is globally well posed is a standard exercise as this problem is energy subcritical [21, 24]. The initial data we consider is described next.

It turns out that the Minkowskian distance  $d_M$  from lemma 1.2.1 is not necessarily defined everywhere. It is, however, defined on the set  $\Sigma_{c, T}$ , defined in (1.14), for some  $c, T > 0$ . For  $0 < \bar{T} \leq T$ , define

$$\mathcal{B} := [0, \bar{T}] \times (R(0) - \delta, R(0) + \delta)$$

where  $\delta > 0$  is chosen so that  $\mathcal{B} \subset \Sigma_{c, T}$ . We choose the initial data of  $\Phi$  as

$$\Phi(0, r) = \begin{cases} (-1, 0) & \text{for } 0 \leq r < b_1 \\ (1, 0) & \text{for } r > b_2 \end{cases} \quad (1.26)$$

$$\partial_t \Phi(0, r) = 0 \text{ for } 0 \leq r < b_1 \text{ and } r > b_2 \quad (1.27)$$

for some  $0 < b_1 < b_2$  to be chosen shortly. Since (1.7) is a wave equation, there is a finite speed of propagation of data. We choose  $b_1$  and  $b_2$  so that

$$\Phi(t, r) = \begin{cases} (-1, 0) & \text{for } 0 \leq r < R(0) - \delta \\ (1, 0) & \text{for } r > R(0) + \delta \end{cases}$$

for all  $0 \leq t \leq \bar{T}$ . Thus, we know the value of  $\Phi$  outside of  $\Sigma_{c, \bar{T}}$ . This reduces the analysis to controlling the error between  $\Phi$  and the right hand side of (1.15) on the region where  $\Phi$  transitions from  $(-1, 0)$  to  $(1, 0)$  - the region  $\Sigma_{c, \bar{T}}$ .

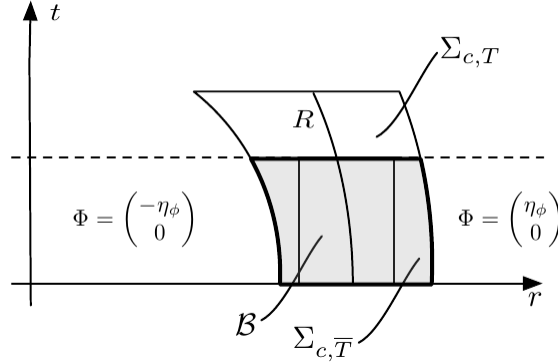


Figure 2:  $\Sigma_{c, T}$  is a neighbourhood on which the Minkowski normal coordinates are well defined. Pick a rectangle  $\mathcal{B} = [0, \bar{T}] \times (R(0) - \delta, R(0) + \delta)$  that lays within  $\Sigma_{c, T}$ . We then pick the initial data of (1.7) so that  $\Phi = (-1, 0)$  to the left of  $\mathcal{B}$  and  $\Phi = (1, 0)$  to the right of  $\mathcal{B}$ .

## 2 Physical Motivation: Superconducting Strings

Motivated by [19], Witten introduced a two-component model, closely related to the abelian-Higgs model, to describe finite energy solutions with vortex filaments supporting superconducting currents [26]. We call this model the **superconducting string model**. It was our initial consideration of this model that lead us to study (1.1) - the superconducting interface model. We will describe what lead us to consider (1.1), but in order to do so we will first need to describe the superconducting interface model.

In [26], an effective action for the superconducting string model using formal arguments was derived. The effective action found suggest that

- (2a) there should be solutions to the model with a vortex filament with a superconducting current
- (2b) the vortex filament is near a codimension two time-like surface  $\Gamma$ , where  $\Gamma$  satisfies a geometric equation that is coupled in a highly nonlinear way to the phase of the current and an ambient vector potential representing an external electromagnetic field

To obtain this effective action, it is proposed that there there exists solutions to the superconducting string model whose profiles to leading order only depend on  $d_M$ . In contrast, the ansatz we use to derive an effective action depends additionally on the gradient of the phase of the field corresponding to the current, see (1.2). The effective action derived in the physics literature in the case when the phase of the current is decoupled from the vector potential looks like the effective action we derived in (A.8) with  $\mu(\gamma^{ij}\partial_i\theta\partial_j\theta)$  replaced with the first order Taylor approximation of  $\mu$  about 0. Results obtained in this paper suggest



that, at least for the superconducting interface model, that the physics ansatz leads to a less accurate approximation of solutions.

To illustrate how the superconducting interface model is related to the superconducting string model, we first need to state the superconducting string model. However, before we can state the superconducting string model we need some notation. We will denote the complex scalar fields as  $\phi, \sigma : \mathbb{R}^{1+3} \rightarrow \mathbb{C}$  and denote their associated gauge fields as  $A_\phi, A_\sigma : \mathbb{R}^{1+3} \rightarrow \mathbb{R}^4$ . We define the covariant derivatives associated to the  $\phi$  and  $\sigma$  fields as  $\nabla_\phi = \nabla - iq_\phi A_\phi$  and  $\nabla_\sigma = \nabla - iq_\sigma A_\sigma$ , respectively, where  $q_\phi, q_\sigma \in \mathbb{R}$  are the coupling constants between  $(\phi, \sigma)$  and their associated gauge fields. As is standard notation, we define  $F_{\phi, \mu\nu} := \partial_\mu A_{\phi, \nu} - \partial_\nu A_{\phi, \mu}$  and similarly define  $F_{\sigma, \mu\nu}$ . Finally, for  $(\lambda_\phi, \lambda_\sigma, \beta) \in \mathbb{R}_+^3$  the **superconducting string potential** is

$$V_S(\phi, \sigma) = \frac{\lambda_\phi}{4}(|\phi|^2 - 1)^2 + \frac{\lambda_\sigma}{4}(|\sigma|^2 - 2)|\sigma|^2 + \frac{\beta}{2}|\phi|^2|\sigma|^2 \quad (2.1)$$

The Lagrangian of the superconducting string model is defined as

$$\mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}\overline{\nabla_{\phi, \alpha}\phi}\nabla_{\phi, \beta}\phi + \frac{1}{2}\eta^{\alpha\beta}\overline{\nabla_{\sigma, \alpha}\sigma}\nabla_{\sigma, \beta}\sigma + \frac{1}{\epsilon^2}V_S(\phi, \sigma) + \frac{\epsilon^2}{4}F_{\phi, \mu\nu}F_{\phi}^{\mu\nu} + \frac{\epsilon^2}{4}F_{\sigma, \mu\nu}F_{\sigma}^{\mu\nu} \quad (2.2)$$

where  $0 < \epsilon \ll 1$  and  $\eta = \text{diag}(-1, 1, 1, 1)$  is the Minkowski metric. An important feature of this model that is worth highlighting is that the  $\sigma$ -field has a  $U(1)$  gauge symmetry. See [25] for an in depth discussion of the physics behind this model.

To obtain (1.1), two changes to the superconducting string model will be made. The first is to consider  $\phi : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ . In this case,  $\phi$  loses its  $U(1)$  gauge symmetry and gains a discrete symmetry. In particular, this allows for  $\phi$  to have an interface. The second change we make is to simplify the problem by decoupling the current from the ambient vector potential. To do this, set  $q_\sigma = 0$ . Applying these changes to (2.2), one obtains the Lagrangian for the superconducting interface model

$$\mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}\eta^{\alpha\beta}\overline{\partial_\alpha\sigma}\partial_\beta\sigma + \frac{1}{\epsilon^2}V_S(\phi, \sigma) \quad (2.3)$$

## 3 Effective Equations

### 3.1 Change to Minkowski Normal Coordinates

Suppose  $\Gamma$  is a codimension 1 time-like surface parameterized by  $\{(y^0, R(y^0))\}$  representing the interface of the  $\phi$ -field we wish to find. Note that (1.7) will determine  $\Gamma$ . It turns out that doing a change of coordinates from polar coordinates to a coordinate system centred about  $\Gamma$  to “straighten out”  $\Gamma$  is quite useful. In fact, it is in these new coordinates that we will find  $\Phi$  which has the properties we want.

The coordinate system we will work in, called **Minkowski normal coordinates** and denoted  $(y^0, y^1)$ , are defined as

$$(t, r) = (y^0, R(y^0)) + y^1\nu(y^0) \quad (3.1)$$

where

$$\eta(\partial_{y^0}(y^0, R(y^0)), \nu(y^0)) = 0 \quad \text{and} \quad \eta(\nu(y^0), \nu(y^0)) = 1$$

We have a choice of  $\nu(y^0)$  and pick

$$\nu(y^0) := \frac{1}{\sqrt{1 - (R')^2}}(R', 1)$$

Going back to lemma 1.2.1 in the introduction,  $y^0$  can be identified with  $s_M$  and  $y^1$  can be identified with  $d_M$ .

The action integral associated to (1.7) is

$$S(\Phi) := \int \left\{ -\frac{1}{2} \partial_t \Phi^2 + \frac{1}{2} \partial_r \Phi^2 + \frac{1}{\epsilon^2} W(\Phi, r) \right\} r dt dr \quad (3.2)$$

where  $\Phi = (\phi, \sigma)$  and where  $W$  is the shifted potential defined in (1.9). Define

$$m := (1 - (R')^2)^{-1/2} \quad \text{and} \quad n := 1 + y^1 m^3 R'' \quad (3.3)$$

A computation shows that

$$\begin{pmatrix} \partial_t \\ \partial_r \end{pmatrix} = \frac{m}{n} \begin{pmatrix} m & -n R' \\ -m R' & n \end{pmatrix} \begin{pmatrix} \partial_{y^0} \\ \partial_{y^1} \end{pmatrix} \quad (3.4)$$

In Minkowski normal coordinates,  $S$  is

$$S(\Phi) = \int \left\{ -\frac{m^2}{2n^2} \partial_{y^0} \Phi^2 + \frac{1}{2} \partial_{y^1} \Phi^2 + \frac{1}{\epsilon^2} W(\Phi, R(y^0) + y^1 m(y^0)) \right\} n(y^0, y^1) (R(y^0) + y^1 m(y^0)) dy^0 dy^1 \quad (3.5)$$

The equations of motion of (3.5) are

$$\frac{m^2}{n^2} \partial_{y^0 y^0} \Phi + B^\alpha \partial_\alpha \Phi - \partial_{y^1 y^1} \Phi + \frac{1}{\epsilon^2} w(\Phi, R(y^0) + y^1 m(y^0)) = 0 \quad (3.6)$$

with initial data as described in section 1.2.1 and where  $w$  was defined in (1.10) and we've defined

$$B^0 := \frac{m}{n} \partial_{y^0} \left( \frac{m}{n} \right) + \frac{1}{(R + y^1 m)} \frac{m^2}{n} R' \quad (3.7)$$

$$B^1 := -\frac{m^3}{n} R'' - \frac{1}{(R + y^1 m)} m \quad (3.8)$$

It turns out that Minkowski normal coordinates are not well defined everywhere. They are, however, well defined on  $[0, y_*^0] \times [-y_*^1, y_*^1]$  where  $y_*^0$  and  $y_*^1$  are determined by the time of existence of  $R$ . We will also choose  $y_*^0$  possibly smaller so that  $R(y^0) \in (r_0, r_1)$  for  $y^0 \in [0, y_*^0]$  where  $r_0 < r_1$  come from the non-degeneracy condition (1.21). Using our choice of initial data, see section 1.2.1,  $y_*^1$  can be chosen possibly smaller so that for  $0 \leq y^0 \leq y_*^0$ , then  $\Phi(y^0, y^1) = (-1, 0)$  for  $y^1 < -y_*^1$  and  $\Phi(y^0, y^1) = (1, 0)$  for  $y^1 > y_*^1$ . Thus, we are left to find solutions to (3.5) on this neighbourhood connecting these two states. For the rest of the paper we consider (3.5).

## 3.2 Expansion

We would like to find a solution to (3.6) which has an interface which is centred on a function  $R(y^0)$ . Furthermore, we'd like the solution to only have an  $O(1)$  change for  $O(\epsilon)$  movements in transverse directions and an  $O(1)$  change for  $O(1)$  movements in tangential directions.

We would like to construct a solution  $\Phi$  of (3.6) as

$$\Phi \approx F_0\left(\frac{y^1}{\epsilon}; R\right) + \epsilon F_1\left(\frac{y^1}{\epsilon}; R, R'\right) \quad (3.9)$$

where  $F_i = (f_i, s_i)$  and each  $F_i$  is independent of  $\epsilon$  (as we discussed following (1.15)). If we plug  $\Phi$  into (3.6) and expand, we can find the equations that each  $F_i$  must satisfy.

Doing so, we find that  $F_0 = F_0(y^1; R)$  must satisfy

$$-\partial_{y^1 y^1} F_0 + w(F_0, R) = 0 \quad (3.10)$$

for each  $R \in \mathbb{R}$  where  $w$  was defined in (1.10). Since  $w(\cdot, R)$  depends on  $R$ , we can see why solutions  $F_0$  depend on  $R$  too.

We also find that for  $v = R'$ , then  $F_1 = F_1(y^1; R, v)$  must satisfy

$$L_1(F_0, R)F_1 = H(R)\partial_{y^1} F_0 - \frac{y^1}{\sqrt{1 - (R')^2}} \partial_R w(F_0, R) \quad (3.11)$$

where

$$L_\epsilon(F_0, R) := -\partial_{y^1 y^1} + \frac{1}{\epsilon^2} \text{Hess}_\Phi W(F_0, R) \quad (3.12)$$

$$H(R) := \frac{1}{\sqrt{1 - (R')^2}} \left( \frac{R''}{1 - (R')^2} + \frac{1}{R} \right) \quad (3.13)$$

$$\partial_R w(\Phi, r) := \lim_{\Delta r \rightarrow 0} \frac{w(\Phi, r + \Delta r) - w(\Phi, r)}{\Delta r} \quad (3.14)$$

The operator  $L_\epsilon(F_0, R)$  is the linearized operator of (3.5), linearized about  $F_0$ , and  $H(R)$  is the **mean curvature** of the surface of rotation generated by  $R$  in  $\mathbb{R}^{1+2}$ . A necessary condition for (3.11) to be solvable is that the right hand side of (3.11) must be orthogonal to the kernel of  $L_\epsilon(F_0, R)$ . This implies that

$$H(R) \int_{\mathbb{R}} \partial_{y^1} F_0(\cdot; R)^2 - \frac{d^2}{R^3} m \int_{\mathbb{R}} s_0(\cdot; R)^2 = 0 \quad (3.15)$$

The following is an important estimate regarding the operator  $L_\epsilon(F_0; R)$  that will be used to verify that there exists a solution to (3.6) satisfying (3.9).

**Theorem 3.2.1** (Spectral Estimate). *Suppose  $F_0$  and  $R$  satisfy (3.10) and (3.15), respectively. By assumption 4 of (1.8),  $\ker(L_\epsilon(F_0, R)) = \text{span}\{\partial_{y^1} F_0\}$ . In particular, this implies that for  $\perp = \perp_{L^2}$ , then for any  $\xi \in \ker(L_\epsilon(F_0, R))^\perp$  we have*

$$\frac{1}{\epsilon^2} \|\xi\|_{L^2(\mathbb{R})}^2 \lesssim \int_{\mathbb{R}} \xi \cdot L_\epsilon(F_0; R) \xi \quad (3.16)$$

Sketch of Proof: For fixed  $R \in (r_0, r_1)$  define

$$X = \{\xi \in H^1(\mathbb{R}; \mathbb{R}^2) : \|\xi\|_2 = 1 \text{ and } \langle \xi, \partial_{y^1} F_0 \rangle_2 = 0\}$$

$$I(\xi) := \int_{\mathbb{R}} \xi \cdot L_\epsilon(F_0, R) \xi$$

To see why (3.16) holds, we want to show that

$$m := \inf_{\xi \in X} I(\xi) > 0$$

Clearly, if  $m \geq \lambda_*$ , where  $\lambda_*$  is from assumption 3 of (1.8), then there is nothing to be done. Suppose  $m < \lambda_*$ . If  $m = 0$  and there exists  $\xi \in X$  at which this infimum is attained then by the non-degeneracy condition (1.21)  $\xi \propto \partial_{y^1} F_0$ . Since  $\xi \in X$ , then  $\xi \perp \partial_{y^1} F_0$  which implies that  $\xi = 0$ . This contradicts the fact that  $\|\xi\|_2 = 1$ . Thus, if we show that there exists  $\xi \in X$  at which the infimum of  $I$  is attained, then we are done.

Let  $\xi_n \in X$  be a minimizing sequence. We have that

$$\|\xi'_n\|_2 \leq I(\xi_n) < C \quad \text{and} \quad \|\xi_n\|_2 = 1 \leq C \quad (3.17)$$

Thus, by possibly passing to a subsequence we have that  $\xi_{n_k} \rightharpoonup \xi$  in  $H^1$ . Furthermore, for  $[f]_{1/2}$  denoting the Holder-1/2 constant of  $f$  we have that

$$[\xi_{n_k}]_{1/2} \leq \|\xi'_n\|_2 \leq I(\xi_n) < C$$

and so  $\xi_{n_k} \rightarrow \xi$  locally uniformly. Thus, we have that

$$I(\xi) \leq \liminf_{n_k \rightarrow \infty} I(\xi_{n_k})$$

Note that we still have that  $\xi \perp \partial_{y^1} F_0$ . Suppose  $\|\xi\|_2 = t \in [0, 1]$ . We have that

$$I(\xi) = t^2 I\left(\frac{\xi}{t}\right) \geq t^2 m$$

Using concentration compactness type arguments one can show that

$$m = \liminf_{n \rightarrow \infty} I(\xi_n) \geq t^2 m + (1 - t^2) \lambda_* \geq m$$

with equality if and only if  $t = 1$ . Thus,  $t = 1$  and so the infimum of  $I$  is attained in  $X$ .

□

### 3.3 Existence of $F_0$ , $F_1$ , and $R$

Set

$$\mu(R) = \inf_{(f,s) \in \mathcal{A}} \int_{\mathbb{R}} \mu(f, s; R) \quad (3.18)$$

$$\mathcal{A} = \left\{ (f, s) \in H^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \mu(f, s; R) < \infty, f(0) = 0, f(\pm\infty) = \pm 1 \right\} \quad (3.19)$$

where

$$\mu(f, s; R) := \frac{1}{2}(f')^2 + \frac{1}{2}(s')^2 + W(f, s; R) \quad (3.20)$$

Notice that without the requirement that  $f(0) = 0$ , then any translation of a minimizer is another minimizer. This condition kills this degeneracy. We will now show that there exists  $(f, s) \in \mathcal{A}$  at which  $\mu(f, s; R)$  attains its infimum.

**Proposition 3.3.1.** *There exists  $(f, s) \in \mathcal{A}$  that solves the minimization problem (3.18).*

Minimization problems like proposition 3.3.1 have been studied extensively. Arguments used in [7] can be used to prove this proposition. In particular, we have the following corollary

**Corollary 3.3.2.** *Set*

$$\tilde{\mathcal{A}} = \{(f, s) \in \mathcal{A} : f(y^1) = 0 \text{ iff } y^1 = 0, |f| \leq 1, 0 \leq s \leq 1, f \text{ is odd}, s \text{ is even}\}$$

*then there exists a minimizer of*

$$\inf_{(f,s) \in \tilde{\mathcal{A}}} \mu(f, s; R)$$

The proof of this corollary follows from the following lemma

**Lemma 3.3.3.**

$$\inf_{(f,s) \in \mathcal{A}} \int_{\mathbb{R}} \mu(f, s; R) = \inf_{(f,s) \in \tilde{\mathcal{A}}} \int_{\mathbb{R}} \mu(f, s; R)$$

The details of this proof are omitted, but the idea is to modify  $(f, s) \in \mathcal{A}$  by using appropriate symmetrizations and translations to show that there is  $(\tilde{f}, \tilde{s}) \in \tilde{\mathcal{A}}$  so that

$$\int_{\mathbb{R}} \mu(\tilde{f}, \tilde{s}; R) \leq \int_{\mathbb{R}} \mu(f, s; R)$$

For notational convenience, we drop the  $\sim$  from  $\tilde{\mathcal{A}}$ .

For each fixed  $R$  corollary 3.3.2 gives us the existence of a minimizer  $F(\cdot; R)$ . One can verify using the non-degeneracy condition (1.21) along with the continuity condition (1.22) that

$$R \mapsto F(\cdot; R)$$

is actually a  $C^2$  map. In fact, for this to be true it suffices that  $F(\cdot; R)$  is a local minimizer, modulo discrete symmetries, for  $R \in (r_1, r_2)$ . Since  $R(0) \in (r_0, r_1)$ , we can then plug  $F(\cdot; R)$  into (3.15) and solve for  $R$  using standard ODE techniques [8] as long as  $R$  remains in  $(r_0, r_1)$ .

**Proposition 3.3.4.** *Define*

$$g(R, v) := \frac{1}{\sqrt{1-v^2}} \frac{d^2}{R^3} \frac{\|s_0(\cdot; R)\|_{L^2(\mathbb{R})}^2}{\|\partial_{y^1} F_0(\cdot; R)\|_{L^2(\mathbb{R})}^2} \quad (3.21)$$

*For every  $(R, v) \in (r_0, r_1) \times (-1, 1)$ , there exists a unique  $F_1 = F_1(y^1; R, v) \in H^1(\mathbb{R}; \mathbb{R}^2)$  solving*

$$L_1(F_0, R)F_1 = g(R, v)\partial_{y^1} F_0(\cdot; R) - \frac{y^1}{\sqrt{1-v^2}} \partial_R w(F_0, R)$$

$$\int F_1 \cdot \partial_{y^1} F_0(\cdot; R) = 0$$

*where  $L_1(F_0, R)$  was defined in (3.12) and  $\partial_R w(F_0, R)$  was defined in (3.14).*

Proof of Proposition 3.3.4: The main tool in proving the existence of  $F_1$  is the spectral theorem [20] applied to the unbounded operator  $L_1(F_0, R) : L^2(\mathbb{R}; \mathbb{R}^2) \rightarrow L^2(\mathbb{R}; \mathbb{R}^2)$ . To use the spectral theorem, we first need to describe the spectrum of  $L_1(F_0, R)$ . The essential spectrum of  $L_1(F_0, R)$  is

$$\sigma_{ess}(L_1(F_0, R)) = [\lambda_*, \infty)$$

where  $\lambda_* > 0$  is from assumption 3 of (1.8). Since  $L_1(F_0, R)$  is self-adjoint, then by the spectral theorem there exists a spectral projection  $E_\lambda$  of  $L_1(F_0, R)$ . Since there exists a spectral gap, by theorem 3.2.1, and since

$$g(R, v)\partial_{y^1}F_0(\cdot; R) + \frac{y^1}{\sqrt{1-v^2}}\partial_R w(F_0, R) \perp \ker(L_1(F_0, R))$$

then  $F_1 \in H^1$  satisfying

$$\langle \psi, F_1 \rangle := \int_{\alpha_0}^{\infty} \frac{1}{\lambda} d \left\langle \psi, E_\lambda \left( g(R, v)\partial_{y^1}F_0(\cdot; R) + \frac{y^1}{\sqrt{1-v^2}}\partial_R w(F_0, R) \right) \right\rangle \quad (3.22)$$

for all  $\psi \in H^1$  solves (3.11), where  $0 < \alpha_0$  is the second smallest value of the spectrum of  $L_1(F_0, R)$ . Moreover, from (3.22), one has that  $F_1 \perp \partial_{y^0}F_0$ .

□

## 3.4 Properties of Profiles

### 3.4.1 Regimes where $s_0$ is nonzero

An important feature of what we are studying is that our model is an interface with a current. For a current to exist, it is necessary that  $s_0 \neq 0$ . For this section we fix the potential to be (2.1). It can easily be checked that for  $\lambda_\sigma < \beta < \lambda_\phi$ , then this potential satisfies all the conditions set out in (1.8), except for condition 4. We believe that (2.1) satisfies this as well, but we did not verify this. We will show that if  $(f_0, s_0)$  satisfies the minimization problem set out in (3.18) for this potential, then  $s_0 \neq 0$ .

Set

$$E(f, s) = \int_{\mathbb{R}} \left\{ \frac{1}{2}(f')^2 + \frac{1}{2}(s')^2 + W(f, s, R) \right\} dy^1$$

For  $s = 0$ , then it is known that  $E_0(f) := E(f, 0)$  is uniquely minimized when

$$f_{min}(y^1) = \tanh\left(\sqrt{\frac{\lambda_\phi}{2}}y^1\right)$$

The goal is to find parameters so that  $E(f_{min}, s) < E(f_{min}, 0)$  for some non-zero  $s$  with  $(f_{min}, s) \in \mathcal{A}$ .

**Proposition 3.4.1.** *If the constants of the model additionally satisfy*

$$\beta < \frac{3}{2}\lambda_\sigma \quad (3.23)$$

*then for sufficiently large  $R$  there exists a minimizer  $(f, s) \in \mathcal{A}$  of (3.18) with  $s \neq 0$ .*

Proof: Note that

$$E(f_{min}, s) = E(f_{min}, 0) + \int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_\sigma}{4}(s^2 - 2)s^2 + \frac{\beta}{2}f^2 s^2 + \frac{d^2}{2R^2}s^2 \right\} \quad (3.24)$$

If we can show that the second term is negative we'd be done. To this end, take

$$s = \frac{1}{\cosh(Bx)}$$

where  $B = \sqrt{\frac{\lambda_\phi}{2}}$ . Thus, plugging  $s$  into the second term we have that

$$\begin{aligned} & \int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_\sigma}{4}(s^2 - 2)s^2 + \frac{\beta}{2}f^2 s^2 + \frac{d^2}{2R^2}s^2 \right\} \\ &= \int_{\mathbb{R}} \left[ \frac{B^2}{2} - \frac{\lambda_\sigma}{4} + \frac{\beta}{2} \right] \frac{\sinh^2(By^1)}{\cosh^4(By^1)} + \int_{\mathbb{R}} \left[ \frac{d^2}{2R^2} - \frac{\lambda_\sigma}{4} \right] \frac{1}{\cosh^2(By^1)} \\ &= \frac{1}{B} \left\{ \frac{\beta}{3} + \frac{d^2}{R^2} - \frac{\lambda}{2} \right\} \end{aligned}$$

The additional constraint (3.23) implies that  $E(f_{min}, s) < E(f_{min}, 0)$  for sufficiently large  $R$ .

□

### 3.4.2 Interface Evolution and Current Quenching

In this section we make two observations about the approximate solution  $F_0$  and the surface  $R$  about which it is concentrated.

The first observation we make is that when the interface we find has a current (i.e.  $s_0 \neq 0$ ), then the interface moves towards the origin. To see why this is true, recall that the surface  $R$  satisfies the geometric relation

$$\frac{1}{\sqrt{1 - (R')^2}} \left( \frac{R''}{1 - (R')^2} + \frac{1}{R} \right) = \frac{1}{\sqrt{1 - (R')^2}} \frac{d^2}{R^3} \frac{\|s_0\|_{L^2(\mathbb{R})}^2}{\|F'_0\|_{L^2(\mathbb{R})}^2}$$

Rearranging, this can be restated as

$$R'' = \left[ \frac{d^2}{R^2} \frac{\|s_0\|_{L^2(\mathbb{R})}^2}{\|F'_0\|_{L^2(\mathbb{R})}^2} - 1 \right] \frac{1 - (R')^2}{R}$$

Since  $F_0$  minimizes  $\mu(f, s; R)$ , then  $F_0$  satisfies

$$-F''_0 + \nabla_\Phi W(F_0, R) = 0$$

Multiplying this by  $F'_0$  and integrating, we also have that  $F_0$  satisfies

$$\frac{1}{2}(F'_0)^2 = W(F_0, R)$$

Thus,

$$R'' = \left[ \frac{1}{2} \frac{d^2}{R^2} \frac{\|s_0\|_{L^2(\mathbb{R})}^2}{\|W(F_0, R)\|_1} - 1 \right] \frac{1 - (R')^2}{R} \quad (3.25)$$

Since

$$W(F_0, R) = V(F_0) + \frac{d^2}{2R^2} s_0^2 > \frac{d^2}{2R^2} s_0^2$$

as  $V(F_0) \geq 0$ , then

$$\frac{1}{2} \frac{d^2}{R^2} \frac{\|s_0\|_{L^2(\mathbb{R})}^2}{\|W(F_0, R)\|_1} > 1$$

and so  $R'' < 0$  (as long as  $|R'| < 1$ ). Since  $R'(0) = 0$ , this implies that  $R$  is moving towards the origin.

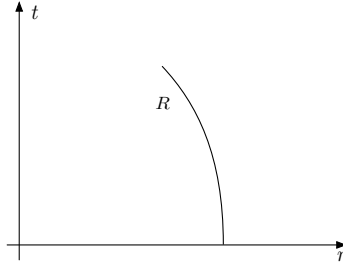


Figure 3: Since  $R'(0) = 0$  and  $R'' < 0$  whenever  $s_0 \neq 0$  then  $R$  is decreasing, at least for a short time.

The second observation we make is that for  $R$  sufficiently small, then for  $F = (f, s)$  minimizing  $\mu(f, s; R)$  we necessarily have that  $s = 0$ . By lemma 3.3.3, for  $(f, s)$  minimizing  $\mu(f, s; R)$ , then  $|f| \leq 1$  and  $|s| \leq 1$ . Then,

$$W(f, s; R) = W(f, 0; R) + \left[ \int_0^1 \partial_\sigma V(f, \lambda s) d\lambda + \frac{d^2}{R^2} \right] s^2$$

By condition 1 of (1.8),  $|\partial_\sigma V(f, \lambda s)| \leq c$  for all  $0 \leq \lambda \leq 1$ . Thus, for sufficiently small  $R > 0$

$$W(f, s; R) > W(f, 0; R)$$

This implies that for sufficiently small  $R$ , then  $(f, s)$  minimizing  $\mu(f, s; R)$  implies that  $s = 0$ .

Suppose  $(f, s)$  minimize  $\mu(f, s; R)$  and suppose we have a potential  $V$  for which there exists a range of  $R$  so that  $s(y^1; R) \neq 0$ . By the second observation, we see that even though  $s(y^1; R) \neq 0$  for some  $R$ , there exists  $R$  sufficiently small for which  $s(y^1; R) = 0$ . Suppose  $R_*$  is the smallest value of  $R$  so that  $s(y^1; R_* + \delta) \neq 0$  for  $0 < \delta \ll 1$ . By the first observation, if  $R_* < R(0) < R_* + \delta$ , then  $R$  becomes smaller as the system evolves. Our solution only makes sense when  $|R'| < 1$ . So the question becomes, does  $R$  become smaller than  $R_*$  before  $|R'| = 1$  and hence do we have current quenching?

Pick  $R_* < R(0) = R_* + \delta$ . As we observed before,  $R$  becomes smaller under the flow of (1.23). Pick  $\delta$  small enough so that

$$-1 \leq \frac{1}{2} \frac{d^2}{R^2} \frac{\|s_0\|_{L^2(\mathbb{R})}^2}{\|W(F_0, R)\|_1} - 1 \leq -\frac{1}{2}$$



for  $\frac{1}{2}R_* \leq R \leq R_* + \delta$  (for  $R < R_*$  this quantity is exactly  $-1$  as  $s = 0$ ). Further, when  $\frac{1}{2}R_* \leq R \leq R_* + \delta$ , then we can estimate (3.25) as

$$-2 \frac{1 - (R')^2}{R_*} \leq R'' \leq -\frac{1}{2} \frac{1 - (R')^2}{R_* + \delta}$$

It follows (after integrating) that

$$-\tanh\left(\frac{4}{R_*}y^0\right) \leq R' \leq -\tanh\left(\frac{4}{R_*+ \delta}y^0\right) \quad (3.26)$$

From this, we can see that  $|R'| < 1$  as long as  $\frac{1}{2}R_* \leq R \leq R_* + \delta$ . Furthermore, we can integrate once more in  $y^0$  to find that

$$(R_* + \delta) - \frac{R_*}{4} \log \cosh\left(\frac{4}{R_*}y^0\right) \leq R \leq (R_* + \delta) \left[1 - \log \cosh\left(\frac{y^0}{R_* + \delta}\right)\right] \quad (3.27)$$

again as long as  $\frac{1}{2}R_* \leq R \leq R_* + \delta$ . From (3.27) and (3.26) we see that for our system  $R$  becomes smaller than  $R_*$  in finite time. Thus, solutions we find undergo current quenching.

### 3.5 Asymptotics of $F_0$ and $F_1$

We would like to examine  $F_0$  for large values of  $y^1$ . This leads us to the following proposition

**Proposition 3.5.1.** *Suppose  $(f, s) \in \mathcal{A}$  solves the minimization problem (3.18). Then there exists  $\alpha > 0$  so that*

$$\begin{cases} 1 - |f| & \lesssim e^{-\alpha|y^1|} \\ |f'| & \lesssim e^{-\alpha|y^1|} \end{cases} \quad \text{and} \quad \begin{cases} |s| & \lesssim e^{-\alpha|y^1|} \\ |s'| & \lesssim e^{-\alpha|y^1|} \end{cases} \quad (3.28)$$

Minimizers  $F = (f, s)$  of (3.18) satisfy  $-F'' + \nabla_\Phi W(F, R) = 0$ . Using assumption 3 of (1.8) one can easily obtain (3.28).

**Proposition 3.5.2.** *Suppose  $F_1$  solves (3.11). Then there exists  $\alpha > 0$  so that for  $\beta = 0, 1$  we have*

$$\left| \partial_{y^1}^\beta F_1 \right| \lesssim e^{-\alpha|y^1|} \quad (3.29)$$

Since the left hand side of (3.11) decays exponentially fast, again using assumption 3 of (1.8) and using standard arguments yields (3.29).

## 4 Effective Dynamics

### 4.1 Approximation Using Profiles Coming From the Formal Asymptotics

The main question we would like to answer is: Suppose  $\Phi$  is a solution to (3.6) with the following properties

- $\Phi = (-1, 0)$  for  $y_1 < -y_*^1$  and  $\Phi = (1, 0)$  for  $y^1 > y_*^1$
- $\Phi(0, y^1)$  is close to  $F_0(\frac{y^1}{\epsilon}; R(0)) + \epsilon F_1(\frac{y^1}{\epsilon}; R(0), R'(0))$
- $\partial_{y^0} \Phi(0, y^1)$  is close to  $\partial_{y^0} \Big|_{y^0=0} (F_0(\frac{y^1}{\epsilon}; R) + \epsilon F_1(\frac{y^1}{\epsilon}; R, R'))$

then does  $\Phi$  remain close to  $F_0(\frac{y^1}{\epsilon}; R) + \epsilon F_1(\frac{y^1}{\epsilon}; R, R')$  up to some  $y_*^0$  independent of  $\epsilon$ ?

We will use the translation symmetry of the profiles  $F_0$  to find a function  $a : [0, y_*^0] \rightarrow \mathbb{R}$  so that for each  $y^0 \in [0, y_*^0]$  the difference between a solution  $\Phi$  to (3.5) and  $F_0(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0))$  is minimized. For each  $R \in \mathbb{R}$  and  $\Psi \in L^2(\mathbb{R}, \mathbb{R}^2)$ , define

$$h_\epsilon(\Psi, R, a) := \left\| \Psi - F_0\left(\frac{y^1 - a}{\epsilon}; R\right) \right\|_{L^2(\mathbb{R})}^2 \quad (4.1)$$

$$G_\epsilon(\Psi, R, a) := \partial_a h_\epsilon(\Psi, R, a) = -\frac{2}{\epsilon} \left\langle \Psi - F_0\left(\frac{y^1 - a}{\epsilon}; R\right), \partial_{y^1} F_0\left(\frac{y^1 - a}{\epsilon}; R\right) \right\rangle_{L^2(\mathbb{R})} \quad (4.2)$$

For each  $y^0 \in [0, y_*^0]$  we want to find a sufficiently regular  $a(y^0)$  so that

$$G_\epsilon(\Phi(y^0), R(y^0), a(y^0)) = 0$$

Define

$$U_{\delta, \epsilon} := \left\{ (\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R} : \inf_{a \in \mathbb{R}} h_\epsilon(\Psi, R, a) < \delta \right\} \quad (4.3)$$

$$V_{\delta, \epsilon}(a_0) := \left\{ (\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R} : h_\epsilon(\Psi, R, a_0) < \delta \right\} \quad (4.4)$$

**Lemma 4.1.1.** *There exists  $\delta > 0$  and a unique  $C^1$  map,  $C^1$  with respect to the  $L^2 \times \mathbb{R}$  topology,  $\tilde{a} : U_{\delta, \epsilon} \rightarrow \mathbb{R}$  so that  $G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0$  where both  $\delta$  and  $\tilde{a}$  possibly depend on  $\epsilon$ .*

Proof: Since

1.  $G_{R, \epsilon}$  is  $C^1$  in  $\Psi$  as it is linear in  $\Psi$ , is  $C^1$  in  $R$  as  $F_0$  is  $C^1$  in  $R$ , and is  $C^1$  in  $a$  because  $F_0$  is  $C^2$  in  $y^1$
2.  $G_\epsilon(F_0(\frac{\cdot - a_0}{\epsilon}; R), R, a_0) = 0$
- 3.

$$\partial_a \Big|_{a=a_0} G_\epsilon(F_0(\frac{\cdot - a_0}{\epsilon}; R), R, a) = \left\langle \partial_{y^1} F_0(\frac{\cdot - a_0}{\epsilon}; R), \partial_{y^1} F_0(\frac{\cdot - a_0}{\epsilon}; R) \right\rangle_{L^2(\mathbb{R})} > 0$$

then we can apply the implicit function. That is, there exists  $\delta > 0$  and a unique  $C^1$  map  $a : V_{\delta, \epsilon}(a_0) \rightarrow \mathbb{R}$ , both  $\delta$  and  $a$  possibly depending on  $\epsilon$ , so that  $G_\epsilon(\Psi, R, a(\Psi, R)) = 0$  for all  $(\Psi, R) \in V_{\delta, \epsilon}(a_0)$ .

Observe that

$$U_{\delta, \epsilon} = \bigcup_{b \in \mathbb{R}} V_{\delta, \epsilon}(a_0 + b)$$

For each  $(\Psi, R) \in U_{\delta, \epsilon}$  there exists  $b \in \mathbb{R}$  so that for  $\tau_b \Psi := \Psi(\cdot - b)$ , then  $(\tau_b \Psi, R) \in V_{\delta, \epsilon}(a_0)$ . Define  $\tilde{a}_b(\Psi, R) = a(\tau_b \Psi, R) + b$ , then  $G_\epsilon(\Psi, R, \tilde{a}_b(\Psi, R)) = 0$ . If  $\tau_c \Psi, \tau_b \Psi \in V_{\delta, \epsilon}(a_0)$ , then by the uniqueness of  $a$  one has that  $\tilde{a}_b(\Psi, R) = \tilde{a}_c(\Psi, R) + c - b$  and thus  $a(\tau_b \Psi, R) = a(\tau_c \Psi, R)$ . Therefore, one can find a unique  $\tilde{a} : U_{\delta, \epsilon} \rightarrow \mathbb{R}$  so that  $G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0$ . □

Suppose  $\Phi$  is a solution to (1.7) and that  $(\Phi(0), R(0)) \in U_{\delta, \epsilon}$  with  $\delta > 0$  coming from lemma 4.1.1. Then, there exists some maximal  $0 < y_*^0(\epsilon)$ , where  $y_*^0(\epsilon)$  may or may not depend on  $\epsilon$  as  $U_{\delta, \epsilon}$  depends on  $\epsilon$ , so that  $(\Phi(y^0), R(y^0)) \in U_{\delta, \epsilon}$  for all  $0 \leq y^0 \leq y_*^0(\epsilon)$ . Thus,

$$G_\epsilon(\Phi(y^0), R(y^0), \tilde{a}(\Phi(y^0), R(y^0))) = 0$$

for  $0 \leq y^0 \leq y_*^0(\epsilon)$ . While proving theorem 1.2.2, we will actually show that  $y_*^0(\epsilon)$  does not depend on  $\epsilon$ . For if  $y_*^0(\epsilon)$  does depend on  $\epsilon$ , then  $(\Phi(y_*^0(\epsilon)), R(y_*^0(\epsilon))) \in U_{\delta, \epsilon}$  which contradicts the maximality of  $y_*^0(\epsilon)$ .

**Corollary 4.1.2.** Suppose  $\Phi$  solves (1.7) and suppose that  $(\Phi(0), R(0)) \in U_{\delta, \epsilon}$  with  $\delta > 0$  from lemma 4.1.1. Then, there exists  $y_*^0(\epsilon) > 0$  and a unique  $C^1$  function  $a(y^0)$  so that  $G_\epsilon(\Phi(y^0), R(y^0), a(y^0)) = 0$  for all  $0 \leq y^0 \leq y_*^0(\epsilon)$ .

Given a solution  $\Phi$  to (1.7), we define

$$\tilde{F}_0(y^0, y^1) = F_0\left(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0)\right) \quad (4.5)$$

where  $a(y^0)$  is from corollary 4.1.2.

Examining (3.11) we see that  $F_1$  does not have a translation symmetry in  $y^1$  as the inhomogeneity of (3.11) depends explicitly on  $y^1$ . Instead, we have that for  $v = R'$ , then  $F_1(\frac{y^1 - a}{\epsilon}; R, v)$  from proposition 3.3.4 solves

$$\epsilon L_\epsilon(\tilde{F}_0; R) \left[ F_1\left(\frac{y^1 - a}{\epsilon}; R, R'\right) \right] = H(R) \partial_{y^1} \tilde{F}_0 - \frac{1}{\epsilon} m(y^0) \left(\frac{y^1 - a}{\epsilon}\right) \partial_R w(\tilde{F}_0, R) \quad (4.6)$$

where  $m$  was defined (3.3). Remember that  $F_1$  was defined independent of  $\epsilon$ . We define

$$\tilde{F}_1(y^0, y^1) = F_1\left(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0), R'(y^0)\right) \quad (4.7)$$

For our result, we'll need to control two quantities. The first is the error and the second is the **shift**  $a(y^0)$ . Define the **error** between  $\Phi$  and our approximation  $\tilde{F}_0 + \epsilon \tilde{F}_1$  as

$$\xi := \Phi - \tilde{F}_0 - \epsilon \tilde{F}_1 \quad (4.8)$$

and define the quantity

$$\underline{A}(y^0) := \left( 1 + \frac{|a(y^0)|}{\epsilon} + \frac{|a'(y^0)|}{\epsilon} \right)^3 \quad (4.9)$$

An observation that we will make use of later is the following. Since  $0 = \partial_a h(a(y^0))$  and  $\tilde{F}_1 \perp \partial_{y^1} \tilde{F}_0$ , we can use (4.2) to get that

$$0 = \int_{\mathbb{R}} \xi \cdot \partial_{y^1} \tilde{F}_0 \quad (4.10)$$

where we needed to use the fact that  $\tilde{F}_1 \perp \partial_{y^1} \tilde{F}_0$ , from proposition 3.3.4, to go from the second line to the third. That is, we have that  $\xi \perp \partial_{y^1} \tilde{F}_0$ .

Next, we will plug  $\Phi = \tilde{F}_0 + \epsilon \tilde{F}_1 + \xi$  into (3.6) and find the equation that  $\xi$  solves. Doing so, we find that  $\xi$  solves

$$\frac{m^2}{n^2} \partial_{y^0 y^0} \xi + B^\alpha \partial_\alpha \xi + L_\epsilon(\tilde{F}_0, R) \xi + S_{-1} + S_0 + N = 0 \quad (4.11)$$

where we used the fact that

$$-\partial_{y^1 y^1} \tilde{F}_0 + \frac{1}{\epsilon^2} w(\tilde{F}_0, R) = 0$$

to simplify and we defined

$$S_{-1} = \epsilon L_\epsilon(\tilde{F}_0, R) \tilde{F}_1 + B^1 \partial_{y^1} \tilde{F}_0 + \frac{y^1}{\epsilon^2} m(y^0) \partial_R w(\tilde{F}_0, R) \quad (4.12)$$

$$S_0 = \frac{m^2}{n^2} \partial_{y^0 y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + B^0 \partial_{y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon B^1 \partial_{y^1} \tilde{F}_1 \quad (4.13)$$

$$N = \frac{1}{\epsilon^2} \left[ w(\tilde{F}_0 + \tilde{F}_\xi, R + y^1 m) - w(\tilde{F}_0, R) - \text{Hess}_\Phi W(\tilde{F}_0, R) \tilde{F}_\xi - y^1 m(y^0) \partial_R w(\tilde{F}_0, R) \right] \quad (4.14)$$

where we've defined  $\tilde{F}_\xi = \epsilon \tilde{F}_1 + \xi$ . Note that (4.11) only makes sense on  $(0, y_*^0) \times (-y_*^1, y_*^1)$ , but since  $\Phi$  and  $\tilde{F}_0 + \epsilon \tilde{F}_\xi$  are both defined on  $(0, y_*^0) \times \mathbb{R}$ , then  $\xi$  is defined on this set too. Recall that we have that  $\Phi = (1, 0)$  for  $y^1 > y_*^1$  and  $\Phi = (-1, 0)$  for  $y^1 < -y_*^1$ . Outside of  $|y^1| \leq y_*^1$ , we use the asymptotics derived in proposition 3.5.1 to get that

$$\|\xi\|_{L^2(|y^1| > y_*^1)} \leq \|(\text{sgn}(y^1), 0) - \tilde{F}_0\|_{L^2(|y^1| > y_*^1)} + \epsilon \|\tilde{F}_1\|_{L^2(|y^1| > y_*^1)} \lesssim e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \quad (4.15)$$

for some  $\alpha > 0$ . Thus, we have that  $\xi$  is small outside of  $|y^1| \leq y_*^1$  if we can control the size of  $a$  and if  $\epsilon$  is taken sufficiently small. We are then left to estimate  $\xi$  on  $|y^1| \leq y_*^1$ . We use the following quantities to control  $\xi$  on  $|y^1| \leq y_*^1$ .

**Definition 4.1.3.** For  $Q = (Q_1, Q_2) : (0, y_*^0) \times (-y_*^1, y_*^1) \rightarrow \mathbb{R}^2$  define the energy density

$$e(Q) = \frac{1}{2} \frac{m^2}{n^2} \partial_{y^0} Q^2 + \frac{1}{2} \partial_{y^1} Q^2 + \frac{1}{2\epsilon^2} Q \cdot \text{Hess}_\Phi W(\tilde{F}_0, R) Q \quad (4.16)$$

Using the energy density, we define the energy of  $Q$  as

$$E(Q) = \int_{|y^1| \leq y_*^1} e(Q) dy^1 \quad (4.17)$$

For convenience we set  $E(y^0) = E(\xi)(y^0)$ .

Using this new definition, we obtain a very useful corollary to theorem 3.2.1 that we will use to control the error term  $\xi$

**Corollary 4.1.4.** Suppose  $\Phi$  is a solution to (3.6) with initial data as described in section 1.2.1. For  $\xi$  as defined in (4.8), we have that

$$\frac{1}{\epsilon^2} \int_{-y_*^1}^{y_*^1} |\xi|^2 \lesssim E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \quad (4.18)$$

for some  $\alpha > 0$

Proof of corollary 4.1.4: Use theorem 3.2.1 along with (4.15) to obtain the estimate (4.18).

□

## 4.2 Main Result

The main theorem of this paper is the following.

**Theorem 4.2.1.** Suppose  $\Phi$  solves (3.6). Further assume that  $\Phi(0)$ ,  $\partial_{y^0} \Phi(0)$ ,  $a(0)$ , and  $a'(0)$  satisfy

$$\underline{A}(0) \lesssim 1 \quad \text{and} \quad E(0) \lesssim \epsilon^2$$

Then there exists  $0 < \bar{y}^0 < y_*^0$ ,  $\bar{y}^0$  independent of  $\epsilon$ , and  $a : (0, \bar{y}^0) \rightarrow \mathbb{R}$  so that

$$\underline{A}(y^0) \lesssim 1 \quad \text{and} \quad E(y^0) \lesssim \epsilon^2$$

for all  $0 \leq y^0 \leq \bar{y}^0$ .

To prove theorem 4.2.1, we will use the following two estimates

**Theorem 4.2.2** (Energy Estimate Theorem). *Suppose  $\Phi$  solves (3.6). Then for as long as  $a(y^0)$  is well defined we have*

$$\begin{aligned} & \left[ 1 - \frac{y_*^1}{(R - y_*^1 m)^3} \right] E - E(0) \\ & \lesssim \left[ \sqrt{\epsilon^3} \sqrt{E} \underline{A} + \sqrt{\epsilon} \sqrt{E^3} + \sqrt{\epsilon} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \underline{A} + \frac{1}{\epsilon} \sqrt{E} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right]_{y^0}^{y^0} + \int_0^{y^0} (E + \epsilon \sqrt{E} + \frac{1}{\epsilon^2} e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \underline{A} \end{aligned} \quad (4.19)$$

**Theorem 4.2.3** (Bounded Shift Theorem). *Suppose  $\Phi$  solves (3.6). Then for as long as  $a(y^0)$  is well defined we have*

$$\left[ 1 - \epsilon \underline{A} - \sqrt{\epsilon} \underline{A} \sqrt{E} - e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right] \frac{|a''|}{\epsilon} \lesssim \frac{1}{\sqrt{\epsilon^5}} (\epsilon \underline{A} + \sqrt{(\epsilon \sqrt{E} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}})^4 \sqrt{E}}) (\sqrt{\epsilon^3} + \epsilon \sqrt{E} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \quad (4.20)$$

Theorem 1.2.2 is obtained from theorem 4.2.1 by applying the spectral estimate (4.18) and using the estimate  $E(y^0) \lesssim \epsilon^2$ . Assuming theorems 4.2.2 and 4.2.3 are true, we can prove theorem 4.2.1.

Proof of Theorem 4.2.1: We will be implementing a bootstrap argument to prove theorem 4.2.1. In order to close the argument to be outlined, we may need to choose  $y_*^0$  and  $y_*^1$  smaller, still independent of  $\epsilon$ , so that

$$C \frac{y_*^1}{(R - y_*^1 m)^3} \leq \frac{1}{2} \quad (4.21)$$

where  $C = C(y_*^0, y_*^1)$  is the constant from theorems 4.2.2 and 4.2.3. We can find such a  $y_*^1$ , because  $C(y_*^0, y_*^1) \rightarrow 0$  as  $y_*^1 \rightarrow 0$ . We'll make use of the following two estimates in order to complete the proof

$$|a| \leq |a(0)| + \int_0^{y^0} |a'| \quad \text{and} \quad |a'| \leq |a'(0)| + \int_0^{y^0} |a''| \quad (4.22)$$

Next, suppose  $a(y^0)$  is well defined on the interval  $I = (0, b)$ . Define

$$E_M(I) := \max_{y^0 \in I} E(y^0) \quad \text{and} \quad \underline{A}_M(I) := \max_{y^0 \in I} \underline{A}(y^0)$$

Using theorem 4.2.2 we have that

$$\begin{aligned} E_M(I) & \lesssim E(0) + \frac{1}{2} E_M(I) + \epsilon^{3/2} \underline{A}_M(I) E_M(I)^{1/2} + \epsilon^{1/2} E_M(I)^{3/2} + \epsilon^{1/2} e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)} \\ & + \frac{1}{\epsilon} E_M(I)^{1/2} e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)} + |I| \left[ E_M(I) + \epsilon E_M(I)^{1/2} + \frac{1}{\epsilon^2} e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)} \right] \underline{A}_M(I) \end{aligned} \quad (4.23)$$

Using theorem 4.2.3 and (4.22) we have that

$$\underline{A}_M(I)^{1/3} \lesssim (1 + |I|) \underline{A}(0)^{1/3} + \frac{|I| + |I|^2}{1 - \epsilon \underline{A}_M(I) - \sqrt{\epsilon} E_M(I)^{1/2} \underline{A}_M(I) + e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)}} \underline{B}_M(I) \quad (4.24)$$

where we've introduced  $\underline{B}_M(I)$  as

$$\underline{B}_M(I) := \frac{1}{\sqrt{\epsilon^5}} \left[ \epsilon \underline{A}_M(I) + \sqrt{\epsilon \sqrt{E_M(I)} + e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)} \sqrt[4]{E_M(I)}} \right] \left[ \sqrt{\epsilon^3} + \epsilon \sqrt{E_M(I)} + e^{-\frac{\alpha}{\epsilon} y_*^1} e^{\alpha \underline{A}_M(I)} \right] \quad (4.25)$$

By corollary 4.1.2,  $a(y^0)$  is well defined up to some time  $\widehat{y}^0$  (corollary 4.1.2 doesn't tell us that  $\widehat{y}^0$  is independent of  $\epsilon$ , just that it exists). There exists  $\bar{y}^0$  so that for  $I = (0, \min\{\widehat{y}^0, \bar{y}^0\})$ , then estimates (4.23) and (4.24) imply that

$$E_M(I) \lesssim \epsilon^2 \quad \text{and} \quad \underline{A}_M(I) \lesssim 1$$

If  $\min\{\widehat{y}^0, \bar{y}^0\} = \bar{y}^0$ , then we are done. If not, then because

$$\underline{E}_M(I) \lesssim \epsilon^2$$

using corollary 4.1.2 we actually have that  $a(y^0)$  exists beyond  $\widehat{y}^0$ . Boot strapping allows us to conclude that  $a(y^0)$  exists and is well defined on  $I = (0, \bar{y}^0)$  and on  $I$  that

$$E_M(I) \lesssim \epsilon^2 \quad \text{and} \quad \underline{A}_M(I) \lesssim 1$$

□

### 4.3 Proof of Energy Estimate (Theorem 4.2.2)

We require an estimate of  $(y^1)^\gamma \partial_{y^1}^\alpha \partial_{y^0}^\beta \tilde{F}_i$ , for  $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$ , to prove this theorem and theorem 4.2.3. A point on notation before continuing. We have that  $F_1 = F_1(y^1; R, v)$  where third slot of  $F_1$  is called  $v$ .

**Lemma 4.3.1.** *For  $\alpha, \beta, \kappa, \lambda$ , and  $\gamma \in \mathbb{N} \cup \{0\}$ , then for  $(R, v) \in (r_0, r_1) \times (-1, 1)$ ,  $r_0 < r_1$  coming from the non-degeneracy condition (1.21), and  $a \in \{a(y^0) : 0 \leq y^0 \leq y_*^0\}$  we have that*

$$\left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial R} \right)^\beta \left[ F_0 \left( \frac{\cdot - a}{\epsilon}; R \right) \right] \right\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}} \left( 1 + \frac{|a(y^0)|}{\epsilon} \right)^\gamma \quad (4.26)$$

$$\left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial R} \right)^\beta \left( \frac{\partial}{\partial v} \right)^\kappa \left[ F_1 \left( \frac{\cdot - a}{\epsilon}; R, v \right) \right] \right\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}} \left( 1 + \frac{|a(y^0)|}{\epsilon} \right)^\gamma \quad (4.27)$$

where the constant in the estimate depends on  $y_*^0$ , but not  $\epsilon$ .

Proof of lemma 4.3.1: For (4.26) we have

$$\begin{aligned} \left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial R} \right)^\beta \left[ F_0 \left( \frac{\cdot - a}{\epsilon}; R \right) \right] \right\|_{L^2(\mathbb{R})} &= \frac{1}{\epsilon^\alpha} \left\| (y^1)^\gamma \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta} \left( \frac{\cdot - a}{\epsilon}; R \right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \frac{1}{\epsilon^{\alpha-\gamma}} \left\| \left( \frac{y^1 - a + a}{\epsilon} \right)^\gamma \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta} \left( \frac{\cdot - a}{\epsilon}; R \right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \frac{1}{\epsilon^{\alpha-\gamma}} \left\| \left[ \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left( \frac{y^1 - a}{\epsilon} \right)^{\gamma-j} \left( \frac{a}{\epsilon} \right)^j \right] \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta} \left( \frac{\cdot - a}{\epsilon}; R \right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \frac{1}{\epsilon^{\alpha-\gamma}} \sum_{j=0}^{\gamma} \left( \frac{|a|}{\epsilon} \right)^j \left\| \left( \frac{y^1 - a}{\epsilon} \right)^{\gamma-j} \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta} \left( \frac{\cdot - a}{\epsilon}; R \right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}} \left( 1 + \frac{|a(y^0)|}{\epsilon} \right)^\gamma \end{aligned}$$

where we did a change of variables and used the exponential decay of  $F_0$  and its derivatives to obtain the last inequality.

We estimate (4.27) in the same way.

□

We will use lemma 4.3.1 to prove the more useful estimates

**Corollary 4.3.2.** *For  $\alpha, \gamma \in \mathbb{N} \cup \{0\}$  and  $\beta = 0, 1, 2$ , then for  $0 \leq y^0 \leq y_*^0$  and for  $i = 1, 2$  we have that*

$$\left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial y^0} \right)^\beta \tilde{F}_i \right\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\epsilon^{\alpha-\gamma-1/2}} (1 + \delta^{\beta 2} \frac{|a''|}{\epsilon}) \left( 1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon} \right)^{\gamma+\beta} \quad (4.28)$$

where  $\delta^{ij}$  is the Kronecker-delta and the constant in the estimate depends on  $y_*^0$ , but not  $\epsilon$ .

Proof of corollary 4.3.2: We will show (4.28) for  $i = 0$ . The same arguments can be used to show (4.28) for  $i = 1$ .

$\beta = 0$ : This directly follows from lemma 4.3.1.

$\beta = 1$ : We have that

$$\frac{\partial}{\partial y^0} \tilde{F}_0 = -\frac{a'}{\epsilon} \frac{\partial F_0}{\partial y^1} + R' \frac{\partial F_0}{\partial R} \quad (4.29)$$

where  $F_0$  and all of its partial derivatives are evaluated at  $(\frac{y^1-a}{\epsilon}; R)$ . We suppressed the arguments of these quantities for notational convenience. Estimating  $(y^1)^\gamma \partial_{y^0} \partial_{y^1}^\alpha \tilde{F}_0$  first we have

$$\begin{aligned} & \left\| (y^1)^\gamma \frac{\partial}{\partial y^0} \left( \frac{\partial}{\partial y^1} \right)^\alpha \tilde{F}_0 \right\|_{L^2(\mathbb{R})} \\ & \leq \left\| (y^1)^\gamma a' \left( \frac{\partial}{\partial y^1} \right)^{\alpha+1} \left[ F_0 \left( \frac{\cdot - a}{\epsilon}; R \right) \right] \right\|_{L^2(\mathbb{R})} + \left\| (y^1)^\gamma R' \left( \frac{\partial}{\partial y^1} \right)^\alpha \frac{\partial}{\partial R} \left[ F_0 \left( \frac{\cdot - a}{\epsilon}; R \right) \right] \right\|_{L^2(\mathbb{R})} \\ & \lesssim \frac{1}{\epsilon^{\alpha-\gamma-1/2}} \left( 1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon} \right)^{\gamma+1} \end{aligned}$$

where we used (4.26) to obtain the last inequality.

$\beta = 2$ : We have that

$$\frac{\partial^2}{\partial (y^0)^2} \tilde{F}_0 = -\frac{a''}{\epsilon} \frac{\partial F_0}{\partial y^1} + \left( \frac{a'}{\epsilon} \right)^2 \frac{\partial^2 F_0}{\partial (y^1)^2} + 2R' \frac{a'}{\epsilon} \frac{\partial^2 F_0}{\partial y^1 \partial R} + (R')^2 \frac{\partial^2 F_0}{\partial R^2} + R'' \frac{\partial F_0}{\partial R} \quad (4.30)$$

where again  $F_0$  and all of its partial derivatives are evaluated at  $(\frac{y^1-a}{\epsilon}; R)$ . Estimating  $(y^1)^\gamma \partial_{y^0}^2 \partial_{y^1}^\alpha \tilde{F}_0$  in the same way as we did when finding (4.28) for  $i = 0$  and  $\beta = 1$ , we have that

$$\left\| (y^1)^\gamma \partial_{y^0}^2 \partial_{y^1}^\alpha \tilde{F}_0 \right\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\epsilon^{\alpha-\gamma-1/2}} \left( 1 + \left| \frac{a''}{\epsilon} \right| \right) \left( 1 + \frac{|a|}{\epsilon} + \left| \frac{a'}{\epsilon} \right| \right)^{\gamma+2}$$

where we used lemma 4.26 to obtain the last inequality.

□

To begin the energy estimate, we will use the following divergence identity.

**Lemma 4.3.3.**

$$\partial_{y^0}\xi \cdot \left[ \frac{m^2}{n^2} \partial_{y^0 y^0} \xi + B^\alpha \partial_\alpha \xi + L_\epsilon(\tilde{F}_0, R) \xi \right] = \operatorname{div}_{y^0, y^1} \vec{X} + Y \quad (4.31)$$

where

$$\begin{aligned} \vec{X} &= (e(\xi), -\partial_{y^0}\xi \cdot \partial_{y^1}\xi) \\ Y &= -\frac{1}{\epsilon^2} \xi \cdot \left[ \partial_{y^0} \operatorname{Hess}_\Phi W(\tilde{F}_0, R) \right] \xi + B^\alpha \partial_{y^0}\xi \cdot \partial_\alpha \xi - \frac{1}{2} \partial_{y^0} \left( \frac{m^2}{n^2} \right) \partial_{y^0} \xi^2 \end{aligned}$$

We omit the proof of lemma 4.3.3 as the proof is a straightforward computation. Using the divergence identity (4.31) and (4.11), we have

$$\partial_{y^0} E = \partial_{y^0} \xi \cdot \partial_{y^1} \xi \Big|_{-y_*^1}^{y_*^1} - \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot [S_{-1} + S_0 + N] - \int_{|y^1| \leq y_*^1} Y$$

Integrating  $\partial_{y^0} E$  with respect to  $y^0$  once to get

$$E(y^0) - E(0) = \int_0^{y^0} \partial_{y^0} \xi \cdot \partial_{y^1} \xi \Big|_{-y_*^1}^{y_*^1} - \int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot [S_{-1} + S_0 + N] - \int_0^{y^0} \int_{|y^1| \leq y_*^1} Y \quad (4.32)$$

This energy identity is the main equation we want to estimate.

We will break the analysis up to simplify things. We will consider each term on the right hand side of (4.32) individually, estimate them, and then in the end add all of the individual estimates back up to obtain the desired estimate.

**Lemma 4.3.4.**

$$- \int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot S_{-1} \lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon A} \Big|_0^{y^0} + \int_0^{y^0} (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon A} \quad (4.33)$$

Proof: Recall that  $\tilde{F}_1$  solves (4.6)

$$\epsilon L_\epsilon(\tilde{F}_0, R) \tilde{F}_1 = H(R) \partial_{y^1} \tilde{F}_0 - \left( \frac{y^1 - a(y^0)}{\epsilon} \right) m(y^0) \partial_{RW}(\tilde{F}_0, R)$$

Using (3.8) and (3.13) we see that

$$B^1 = -H(R) + O(y^1)$$

Recall the definition of  $S_{-1}$  (4.12)

$$S_{-1} = \left[ H(R) + B^1 \right] \partial_{y^1} \tilde{F}_0 + \frac{a}{\epsilon} m(y^0) \partial_{RW}(\tilde{F}_0, R) \quad (4.34)$$



Integrating by parts in  $y^0$  we have

$$-\int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot S_{-1} = - \int_{|y^1| \leq y_*^1} \xi \cdot S_{-1} \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y_*^1} \xi \cdot \partial_{y^0} S_{-1} \quad (4.35)$$

For  $j = 0, 1$  we will need to estimate

$$\int_{-y_*^1}^{y_*^1} \xi \cdot \partial_{y^0}^j S_{-1} \lesssim \|\xi\|_{L^2(-y_*^1, y_*^1)} \left\| \partial_{y^0}^j S_{-1} \right\|_{L^2(\mathbb{R})} \lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \left\| \partial_{y^0}^j S_{-1} \right\|_{L^2(\mathbb{R})} \quad (4.36)$$

We estimate  $\|S_{-1}\|_{L^2(\mathbb{R})}$  first as

$$\|S_{-1}\|_{L^2(\mathbb{R})} \lesssim \|(y^1) \partial_{y^1} \tilde{F}_0\|_{L^2(\mathbb{R})} + \left\| \frac{a(y^0)}{\epsilon} m(y^0) \partial_R w(\tilde{F}_0, R) \right\|_{L^2(\mathbb{R})} \lesssim \sqrt{\epsilon} \underline{A} \quad (4.37)$$

where we used corollary 4.3.2 to obtain the last inequality. We estimate the second term of (4.35) as

$$\|\partial_{y^0} S_{-1}\|_{L^2(\mathbb{R})} \lesssim \left\| \partial_{y^0} \left[ (H(R) + B^1) \partial_{y^1} \tilde{F}_0 \right] \right\|_{L^2(-y_*^1, y_*^1)} + \left\| \partial_{y^0} \left[ \frac{a}{\epsilon} m(y^0) \partial_R w(\tilde{F}_0, R) \right] \right\|_{L^2(-y_*^1, y_*^1)} \lesssim \sqrt{\epsilon} \underline{A} \quad (4.38)$$

where we again used corollary 4.3.2 to obtain the last inequality. Combining (4.36), (4.37), and (4.38) finishes the proof.  $\square$

**Lemma 4.3.5.**

$$-\int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot S_0 \lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A} \Big|_0^{y^0} + \int_0^{y^0} (1 + \frac{|a''|}{\epsilon}) (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A} \quad (4.39)$$

Proof: Using the definition of  $S_0$ , see (4.13), the left hand side of (4.39) is

$$-\int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot \left[ \frac{m^2}{n^2} \partial_{y^0 y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + B^0 \partial_{y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon B^1 \partial_{y^1} \tilde{F}_1 \right]$$

We would like to integrate by parts in  $y^0$  to move the derivative from  $\xi$  to  $S_0$  and use corollary 4.3.2. However, then we'd have to estimate  $\partial_{y^0 y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1)$  which will give rise to  $a'''$  terms which we'd rather avoid. So, we take special care when estimating these two problematic terms and proceed as we would like to for the other terms.

1.  $\partial_{y^0 y^0} \tilde{F}_0$  term: Recall that  $\tilde{F}_0 = F_0(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0))$  and so

$$\partial_{y^0 y^0} \tilde{F}_0 = \frac{a''}{\epsilon} \partial_{y^1} F_0 - \left(\frac{a'}{\epsilon}\right)^2 \partial_{y^1 y^1} F_0 + 2R' \frac{a'}{\epsilon} \partial_R \partial_{y^1} F_0 - R'' \partial_R F_0 - (R')^2 \partial_{RR} F_0 \quad (4.40)$$

where we use the notation  $\partial_{y^1}^\alpha \partial_R^\beta F_0 = \partial_{y^1}^\alpha \partial_R^\beta F_0(\frac{y^1-a}{\epsilon}; R)$  to simplify things. To control the  $a''$  term, we use the fact that  $\xi \perp \partial_{y^1} F_0$ . Differentiating

$$0 = \int_{\mathbb{R}} \xi \cdot \partial_{y^1} \tilde{F}_0$$

with respect to  $y^0$  once yields

$$\int_{\mathbb{R}} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 = \frac{a'}{\epsilon} \int_{\mathbb{R}} \xi \cdot \partial_{y^1 y^1} F_0 - R' \int_{\mathbb{R}} \xi \cdot \partial_R \partial_{y^1} F_0 \quad (4.41)$$

To use this we first recall that  $m = (1 - (R')^2)^{-1/2}$  and  $n = 1 + y^1 m^3 R''$ . Thus,

$$\int_0^{y^0} \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 = \int_0^{y^0} \int_{|y^1| \leq y_*^1} m^2 \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 + \int_0^{y^0} \int_{|y^1| \leq y_*^1} \left[ \frac{m^2}{n^2} - m^2 \right] \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0$$

Using (4.41) we control the first term, the  $m^2$  term, as follows

$$\begin{aligned} \left| \int_0^{y^0} \int_{|y^1| \leq y_*^1} m^2 \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 \right| &\lesssim \int_0^{y^0} |m|^2 \frac{|a''|}{\epsilon} \left| \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 \right| \\ &\lesssim \int_0^{y^0} \left| \frac{a' a''}{\epsilon^2} \right| \int_{\mathbb{R}} |\xi \cdot \partial_{y^1 y^1} F_0| + \int_0^{y^0} \left| \frac{a''}{\epsilon} \right| \int_{\mathbb{R}} |\xi \cdot \partial_R \partial_{y^1} F_0| \end{aligned}$$

where we used the fact that  $|m|^2 \lesssim 1$  for all  $y^0 \leq y_*^0$  to go from the 1st line to the 2nd. Remember that we are using the notation  $\partial_{y^1}^\alpha \partial_R^\beta F_0 = \partial_{y^1}^\alpha \partial_R^\beta F_0(\frac{y^1-a}{\epsilon}; R)$ . We then use the Cauchy-Schwarz inequality, corollary 4.3.2, lemma 3.5.1, and (4.18) to conclude that

$$\left| \int_0^{y^0} \int_{|y^1| \leq y_*^1} m^2 \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 \right| \lesssim \int_0^{y^0} \frac{|a''|}{\epsilon} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1-a}{\epsilon}}) \sqrt{\epsilon} \underline{A}$$

To control the second term (i.e. the  $\frac{m^2}{n^2} - m^2$  term), observe that  $\frac{m^2}{n^2} - m^2 = O(y^1)$ . We use the Cauchy-Schwarz inequality and the definition of energy to get

$$\begin{aligned} \left| \int_0^{y^0} \int_{|y^1| \leq y_*^1} \left[ \frac{m^2}{n^2} - m^2 \right] \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 \right| &\lesssim \int_0^{y^0} \frac{|a''|}{\epsilon} \left\| \partial_{y^0} \xi \right\|_{L^2(-y_*^1, y_*^1)} \left\| y^1 \partial_{y^1} F_0 \right\|_{L^2(\mathbb{R})} \\ &\lesssim \int_0^{y^0} \frac{|a''|}{\epsilon} \epsilon^{3/2} E^{1/2} \underline{A} \end{aligned}$$

To summarize, we have the following estimate

$$\int_0^{y^0} \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \frac{a''}{\epsilon} \partial_{y^0} \xi \cdot \partial_{y^1} F_0 \lesssim \int_0^{y^0} \frac{|a''|}{\epsilon} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1-a}{\epsilon}}) \sqrt{\epsilon} \underline{A}$$

We deal with the rest of the terms of (4.40) by shifting the  $\partial_{y^0}$  off of  $\xi$  and use (4.18) along with (4.3.1). We estimate the  $\partial_{y^1 y^1} F_0$  term of (4.40) and the estimation of the other three terms of (4.40) are done in the same way yielding the same bounds. To this end, we estimate the  $\partial_{y^1 y^1} F_0$  term as

$$\begin{aligned}
& \int_0^{y^0} \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \partial_{y^0} \xi \cdot \left[ -\left(\frac{a'}{\epsilon}\right)^2 \partial_{y^1 y^1} F_0 \right] \\
&= - \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \left(\frac{a'}{\epsilon}\right)^2 \xi \cdot \partial_{y^1 y^1} F_0 \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y_*^1} \xi \cdot \partial_{y^0} \left[ \frac{m^2}{n^2} \left(\frac{a'}{\epsilon}\right)^2 \partial_{y^1 y^1} F_0 \right] \\
&\lesssim \left(\frac{a'}{\epsilon}\right)^2 \|\xi\|_{L^2(-y_*^1, y_*^1)} \|\partial_{y^1 y^1} F_0\|_{L^2(\mathbb{R})} \Big|_0^{y^0} + \int_0^{y^0} \|\xi\|_{L^2(-y_*^1, y_*^1)} \left\| \partial_{y^0} \left[ \frac{m^2}{n^2} \left(\frac{a'}{\epsilon}\right)^2 \partial_{y^1 y^1} F_0 \right] \right\|_{L^2(\mathbb{R})} \\
&\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \Big|_0^{y^0} + \int_0^{y^0} (1 + \frac{|a''|}{\epsilon}) (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}}
\end{aligned}$$

Putting together the above estimates yields

$$- \int_0^{y^0} \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \partial_{y^0} \xi \cdot \partial_{y^0 y^0} \tilde{F}_0 \lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \Big|_0^{y^0} + \int_0^{y^0} (1 + \frac{|a''|}{\epsilon}) (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \quad (4.42)$$

2.  $\partial_{y^0 y^0} \tilde{F}_1$  term: Since  $\frac{m^2}{n^2} \lesssim 1$  on  $(0, y_*^0) \times (-y_*^1, y_*^1)$ , then after using Cauchy-Schwarz and corollary 4.3.2 we have

$$- \epsilon \int_0^{y^0} \int_{|y^1| \leq y_*^1} \frac{m^2}{n^2} \partial_{y^0} \xi \cdot \partial_{y^0 y^0} \tilde{F}_1 = \epsilon \int_0^{y^0} E^{1/2} \|\partial_{y^0 y^0} \tilde{F}_1\|_{L^2(\mathbb{R})} \lesssim \int_0^{y^0} (1 + \frac{|a''|}{\epsilon}) \epsilon^{3/2} E^{1/2} \underline{A} \quad (4.43)$$

3.  $B^0 \partial_{y^0} \tilde{F}_0$ : Using the boundedness of  $B^0$  on  $(0, y_*^0) \times (-y_*^1, y_*^1)$  (defined in (3.7)), we have

$$\begin{aligned}
& - \int_0^{y^0} \int_{|y^1| \leq y_*^1} B^0 \partial_{y^0} \xi \cdot \partial_{y^0} \tilde{F}_0 \\
&= - \int_{|y^1| \leq y_*^1} B^0 \xi \cdot \partial_{y^0} \tilde{F}_0 \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y_*^1} \xi \cdot \partial_{y^0} [B^0 \partial_{y^0} \tilde{F}_0] \\
&\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \Big|_0^{y^0} + \int_0^{y^0} (1 + \frac{|a''|}{\epsilon}) (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \quad (4.44)
\end{aligned}$$

where we used corollary 4.3.2 to obtain the last inequality.

4.  $B^0 \partial_{y^0} \tilde{F}_1$  term: We estimate this term in the same way as the  $B^0 \partial_{y^0} \tilde{F}_0$  term. That is,

$$- \epsilon \int_0^{y^0} \int_{-y_*^1}^{y_*^1} B^0 \partial_{y^0} \xi \cdot \partial_{y^0} \tilde{F}_1 \lesssim \int_0^{y^0} (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \quad (4.45)$$

5.  $B^1 \partial_{y^1} \tilde{F}_1$  term: Using the boundedness of  $B^1$  on  $(0, y_*^0) \times (-y_*^1, y_*^1)$  we get

$$\begin{aligned} & - \int_0^{y^0} \int_{|y^1| \leq y_*^1} \epsilon B^1 \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_1 \\ &= - \int_{|y^1| \leq y_*^1} \xi \cdot [\epsilon B^1 \partial_{y^1} \tilde{F}_1] \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y_*^1} \xi \cdot \partial_{y^0} [\epsilon B^1 \partial_{y^1} \tilde{F}_1] \\ &\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \Big|_0^{y^0} + \int_0^{y^0} (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \sqrt{\epsilon \underline{A}} \end{aligned} \quad (4.46)$$

where we used corollary 4.3.2 to obtain the last inequality.

Putting together the estimates obtained from steps 1-5 we get the desired estimate.

□

**Lemma 4.3.6.**

$$\left| \int_0^{y^0} \int_{|y^1| \leq y_*^1} \partial_{y^0} \xi \cdot N \right| \lesssim \left[ \frac{y_*^1}{(R - y_*^1 m)^3} E + \left( \sqrt{\epsilon^3 \underline{A}} + \frac{1}{\epsilon} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right) \sqrt{E} + \sqrt{\epsilon} \sqrt{E^3} \right] \Big|_0^{y^0} + \int_0^{y^0} \epsilon \sqrt{E} \quad (4.47)$$

Note that we have a factor  $\frac{y_*^1}{(R - y_*^1 m)^3}$  in the above estimate. We'll need to pick  $y_*^1$  sufficiently small so that the constant coming from this estimate multiplied by these two factors is less than 1. We need this to be able to close the bootstrap argument.

Proof: Recalling (4.14), we have

$$N = -\frac{1}{\epsilon^2} \left[ w(\tilde{F}_0 + \tilde{F}_\xi, R + y^1 m) - w(\tilde{F}_0, R) - \text{Hess}_\Phi(\tilde{F}_0, R) \tilde{F}_\xi - y^1 m (y^0) \partial_R w(\tilde{F}_0, R) \right]$$

where, recall,  $\tilde{F}_\xi = \epsilon \tilde{F}_1 + \xi$ . Using the identity  $g(t) = g(0) + g'(0)t + \int_0^1 (1-t)g''(t)dt$  we can rewrite  $N$  as

$$N = -\frac{1}{\epsilon^2} \int_0^1 (1-t) \frac{d^2}{dt^2} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) dt$$

Examining (4.47) and integrating by parts with respect to  $y^0$  one has that

$$-\int_0^{y^0} \int_{-y_*^1}^{y_*^1} \partial_{y^0} \xi \cdot N = -\int_{-y_*^1}^{y_*^1} \xi \cdot N \Big|_0^{y^0} + \int_0^{y^0} \int_{-y_*^1}^{y_*^1} \xi \cdot \partial_{y^0} N$$

Using Cauchy-Schwarz, we have

$$-\int_0^{y^0} \int_{-y_*^1}^{y_*^1} \partial_{y^0} \xi \cdot N \lesssim \epsilon E^{1/2} \|N\|_{L^2(-y_*^1, y_*^1)} \Big|_0^{y^0} + \int_0^{y^0} \epsilon E^{1/2} \|\partial_{y^0} N\|_{L^2(-y_*^1, y_*^1)} \quad (4.48)$$

We are then left to estimate  $\|N\|_{L^2(-y_*^1, y_*^1)}$  and  $\|\partial_{y^0} N\|_{L^2(-y_*^1, y_*^1)}$ .

Before that, we examine the  $\frac{d^2}{dt^2}$  term in  $N$ . Recall that  $\tilde{F}_\xi(y^0, y^1)$  is a two-component vector. For  $\tilde{F}_\xi = ((\tilde{F}_\xi)_\phi, (\tilde{F}_\xi)_\sigma)$ , we have that

$$\begin{aligned} & \frac{d^2}{dt^2} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \\ &= ((\tilde{F}_\xi)_\phi \partial_\phi + (\tilde{F}_\xi)_\sigma \partial_\sigma) \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \\ & \quad + 2y^1 m(y^0) \partial_R \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi + (y^1 m)^2 \partial_{RR} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \end{aligned}$$

$\|\partial_{y^0} N\|_{L^2(-y_*^1, y_*^1)}$  Estimate: For the purpose of our result we only need  $\|\partial_{y^0} N\|_{L^2(-y_*^1, y_*^1)} \lesssim 1$ . This is straightforward as  $\tilde{F}_0$  and  $\tilde{F}_\xi$ , and all time derivatives of these quantities have bounded  $L^2$ -norm.

$\|N\|_{L^2(-y_*^1, y_*^1)}$  Estimate:

1.  $(\tilde{F}_\xi)_\phi$  term:

$$\left\| \int_0^1 (1-t) (\tilde{F}_\xi)_\phi \partial_\phi \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi dt \right\|_{L^2(-y_*^1, y_*^1)} \lesssim \|(\tilde{F}_\xi)_\phi\|_\infty \|\tilde{F}_\xi\|_{L^2(-y_*^1, y_*^1)}$$

where we used the boundedness of  $\tilde{F}_\xi$  to control the operator norm  $\|\partial_\phi \text{Hess}_\Phi W\|$ . Next, we use Gagliardo-Nirenberg to show that

$$\begin{aligned} \|\tilde{F}_\xi\|_\infty &\lesssim \epsilon \|\tilde{F}_1\|_\infty + \|\xi\|_\infty \\ &\lesssim \epsilon + \|\xi\|_{L^2(-y_*^1, y_*^1)}^{1/2} \|\partial_{y^1} \xi\|_{L^2(-y_*^1, y_*^1)}^{1/2} \\ &\lesssim \epsilon + \epsilon^{1/2} E^{1/2} \end{aligned}$$

and we estimate

$$\begin{aligned} \|\tilde{F}_\xi\|_{L^2(-y_*^1, y_*^1)} &\lesssim \epsilon \|\tilde{F}_1\|_{L^2(\mathbb{R})} + \|\xi\|_{L^2(-y_*^1, y_*^1)} \\ &\lesssim \epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) \end{aligned}$$

Putting these together, we obtain the estimate

$$\begin{aligned} & \left\| -\frac{1}{\epsilon^2} \int_0^1 (1-t)(\tilde{F}_\xi)_\phi \partial_\phi \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi dt \right\|_{L^2(-y_*^1, y_*^1)} \\ & \lesssim \epsilon^{1/2} + (1 + \frac{1}{\epsilon^{3/2}} e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) E^{1/2} + \frac{1}{\sqrt{\epsilon}} E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \end{aligned}$$

2.  $(\tilde{F}_\xi)_\sigma$  term: We estimate this term in the same way we did the first. Thus,

$$\begin{aligned} & \left\| -\frac{1}{\epsilon^2} \int_0^1 (1-t)(\tilde{F}_\xi)_\sigma \partial_\sigma \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi dt \right\|_{L^2(-y_*^1, y_*^1)} \\ & \lesssim \epsilon^{1/2} + (1 + \frac{1}{\epsilon^{3/2}} e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) E^{1/2} + \frac{1}{\sqrt{\epsilon}} E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \end{aligned}$$

3.  $2y^1 m(y^0) \partial_R \text{Hess}_\Phi W \tilde{F}_\xi$  term:

$$\begin{aligned} & \left\| -\frac{1}{\epsilon^2} \int_0^1 (1-t) 2y^1 m \partial_R \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi dt \right\|_{L^2(-y_*^1, y_*^1)} \\ & \lesssim \frac{1}{\epsilon^2} \left\| \frac{y^1}{(R - y_*^1 m)^3} \tilde{F}_\xi \right\|_{L^2(-y_*^1, y_*^1)} \\ & \lesssim \frac{1}{\epsilon^2} \left( \epsilon^{5/2} \underline{A} + \epsilon \frac{y_*^1}{(R - y_*^1 m)^3} E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right) \end{aligned}$$

where we made use of corollary 4.3.2 to obtain the estimate.

4.  $(y^1 m(y^0))^2 \partial_{RR} W$  term: Since

$$\partial_{RR} W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) = \begin{pmatrix} 0 \\ \frac{3d^2}{(R + ty^1 m)^4} (\tilde{s}_0 + t(\tilde{F}_\xi)_\sigma) \end{pmatrix}$$

then we have

$$\begin{aligned} & \left\| -\frac{1}{\epsilon^2} \int_0^1 (1-t) (y^1 m)^2 \partial_{RR} W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) dt \right\|_{L^2(-y_*^1, y_*^1)} \\ & \lesssim \frac{1}{\epsilon^2} \left( \|(y^1)^2 \tilde{s}_0\|_{L^2(\mathbb{R})} + \epsilon \|(y^1)^2 \tilde{s}_1\|_{L^2(\mathbb{R})} + \left\| \frac{(y^1)^2}{(R - y^1 m)^4} \xi \right\|_{L^2(-y_*^1, y_*^1)} \right) \\ & \lesssim \frac{1}{\epsilon^2} \left( \epsilon^{5/2} \underline{A} + \epsilon^{7/2} \underline{A} + \epsilon \frac{(y_*^1)}{(R - y_*^1 m)^3} E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right) \end{aligned}$$

where again we made use of corollary 4.3.2 to obtain the estimate.

Putting together estimate (4.48) and the estimates from steps 1-4, we obtain (4.47).

□

**Lemma 4.3.7.**

$$-\int_0^{y^0} \int_{-y_*^1}^{y_*^1} Y \lesssim \int_0^{y^0} \left( E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y^1 - a}{\epsilon}} \right) \quad (4.49)$$

Proof: (See lemma 4.31 for the definition of  $Y$ )

$$\begin{aligned} -\int_0^{y^0} \int_{-y_*^1}^{y_*^1} Y &= \frac{1}{\epsilon^2} \int_0^{y^0} \int_{-y_*^1}^{y_*^1} \xi \cdot [\partial_{y^0} \text{Hess}_\Phi W(\tilde{F}_0, R)] \xi - \int_0^{y^0} \int_{|y^1| \leq y_*^1} B^\alpha \partial_{y^0} \xi \cdot \partial_\alpha \xi - \int_0^{y^0} \int_{-y_*^1}^{y_*^1} \frac{1}{2} \partial_{y^0} \left( \frac{m^2}{n^2} \right) \partial_{y^0} \xi^2 \\ &\lesssim \int_0^{y^0} \left( E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y^1 - a}{\epsilon}} \right) \end{aligned}$$

where we used the boundedness of the operator  $\partial_{y^0} \text{Hess}_\Phi W(\tilde{F}_0, R)$  and the  $B^\alpha$ 's on  $(0, y_*^0) \times (-y_*^1, y_*^1)$  and (4.18) to obtain the estimate.

□

**Lemma 4.3.8.**

$$\int_0^{y^0} \partial_{y^0} \xi(y^0, \pm y_*^1) \cdot \partial_{y^1} \xi(y^0, \pm y_*^1) \lesssim \int_0^{y^0} \frac{1}{\epsilon} e^{-\alpha \frac{y^1 - a}{\epsilon}} \underline{A} \quad (4.50)$$

Proof: Recall that

$$\xi(y^0, \pm y_*^1) = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} - \tilde{F}_0(y^0, \pm y_*^1) - \epsilon \tilde{F}_1(y^0, \pm y_*^1)$$

Differentiating  $\tilde{F}_0$  and  $\tilde{F}_1$  with respect to  $y^0$  and using (3.5.1), then

$$|\partial_{y^0} \xi(y^0, \pm y_*^1)| \lesssim e^{-\alpha \frac{y_*^1 - a(y^0)}{\epsilon}} \underline{A}$$

Using (3.5.1), we also have that

$$\left| \partial_{y^1} \xi(y^0, \pm y_*^1) \right| = \left| -\frac{1}{\epsilon} \partial_{y^1} F_0 \left( \frac{y_*^1 - a(y^0)}{\epsilon}; R(y^0) \right) - \partial_{y^1} F_1 \left( \frac{y_*^1 - a(y^0)}{\epsilon}; R(y^0), R'(y^0) \right) \right| \lesssim \frac{1}{\epsilon} e^{-\alpha \frac{y_*^1 - a(y^0)}{\epsilon}}$$

□

Combining the estimates obtained from lemma 4.3.4 to lemma 4.3.8 allows us to conclude the proof of theorem 4.2.2.

#### 4.4 Proof of Bounded Shift Theorem (Theorem 4.2.3)

To prove this we will use the fact that  $\xi \perp \partial_{y^1} \tilde{F}_0$ . Differentiate this quantity with respect to  $y^0$  twice to get

$$\begin{aligned}
0 &= \partial_{y^0 y^0} \int_{\mathbb{R}} \xi \cdot \partial_{y^1} \tilde{F}_0 \\
&= \int_{\mathbb{R}} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \int_{\mathbb{R}} \partial_{y^0} \xi \cdot \partial_{y^0 y^1} \tilde{F}_0 + \int_{\mathbb{R}} \xi \cdot \partial_{y^0 y^0} \partial_{y^1} \tilde{F}_0 \\
&= \int_{\mathbb{R}} (\partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0 y^0} \tilde{F}_0) \\
\int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 &= \int_{\mathbb{R}} (2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0 y^0} \tilde{F}_0) - \int_{|y^1| > y_*^1} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 \quad (4.51)
\end{aligned}$$

where we integrated by parts to move the  $\partial_{y^1}$  off of the  $\partial_{y^0 y^0} \partial_{y^1} \tilde{F}_0$  term onto the  $\xi$  term to obtain the second last equality. On the other hand, we can use the equation for  $\xi$  (4.11) to rewrite the left hand side of (4.51) as

$$\int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 = - \int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2} [B^\alpha \partial_\alpha \xi + L_\epsilon(\tilde{F}_0, R) \xi + S_{-1} + S_0 + N] \cdot \partial_{y^1} \tilde{F}_0 \quad (4.52)$$

Examining the  $S_0$  term on the right hand side of (4.52) more closely, we see that

$$\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^1} \tilde{F}_0 = \int_{-y_*^1}^{y_*^1} \left( \partial_{y^0 y^0} \tilde{F}_0 + \epsilon \partial_{y^0 y^0} \tilde{F}_1 + \frac{n^2}{m^2} \partial_{y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon \frac{n^2}{m^2} B^1 \partial_{y^1} \tilde{F}_1 \right) \cdot \partial_{y^1} \tilde{F}_0 \quad (4.53)$$

Next, examine the term containing  $\partial_{y^0 y^0} \tilde{F}_0$ . Using  $\tilde{F}_0(y^0, y^1) = F_0(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0))$ , the definition of  $\tilde{F}_0$ , to see that

$$\begin{aligned}
\int_{-y_*^1}^{y_*^1} \partial_{y^0 y^0} \tilde{F}_0 \cdot \partial_{y^1} \tilde{F}_0 &= -\frac{a''}{\epsilon^2} \int_{-y_*^1}^{y_*^1} \partial_{y^1} F_0^2 + \frac{1}{\epsilon} \int_{-y_*^1}^{y_*^1} \left[ \left( \frac{a'}{\epsilon} \right)^2 \partial_{y^1 y^1} F_0 - 2R' \frac{a'}{\epsilon} \partial_{y^1} \partial_R F_0 \right. \\
&\quad \left. + R'' \partial_R F_0 + (R')^2 \partial_{RR} F_0 \right] \cdot \partial_{y^1} F_0 \quad (4.54)
\end{aligned}$$

where again we've used the notation  $\partial_{y^1}^\alpha \partial_R^\beta F_i = \partial_{y^1}^\alpha \partial_R^\beta F_i(\frac{y^1 - a}{\epsilon}; R)$ . We would like to obtain a bound for  $\left| \frac{a''}{\epsilon} \right|$  in order to control  $\underline{A}$ . To do this, we will use (4.51 - 4.54) and isolate for the  $\partial_{y^1} F_0^2$  term. We will then use this expression to obtain theorem 4.2.3.

1. Using (4.54), Cauchy-Schwarz, and corollary 4.3.2 we have

$$\frac{|a''|}{\epsilon} \int_{|y^1| \leq y_*^1} \partial_{y^1} F_0^2 \lesssim \left| \int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \tilde{F}_0 \cdot \partial_{y^1} \tilde{F}_0 \right| + \underline{A}$$



2. Using (4.53), Cauchy-Schwarz, and corollary 4.3.2 we have

$$\left| \int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \tilde{F}_0 \cdot \partial_{y^1} \tilde{F}_0 \right| \lesssim \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^1} \tilde{F}_0 \right| + (1 + |a''|) \underline{A}$$

3. Using (4.52), Cauchy-Schwarz, and corollary 4.3.2 we have

$$\begin{aligned} \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^1} \tilde{F}_0 \right| &\lesssim \left| \int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 \right| + \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} B^\alpha \partial_\alpha \xi \cdot \partial_{y^1} \tilde{F}_0 \right| \\ &+ \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} L_\epsilon(\tilde{F}_0, R) \xi \cdot \partial_{y^1} \tilde{F}_0 \right| + \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} S_{-1} \cdot \partial_{y^1} \tilde{F}_0 \right| + \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} N \cdot \partial_{y^1} \tilde{F}_0 \right| \end{aligned}$$

We will estimate the  $B^\alpha \partial_\alpha \xi$ ,  $L_\epsilon(\tilde{F}_0, R) \xi$ ,  $S_{-1}$ , and  $N$  terms separately.

(a)  $B^\alpha \partial_\alpha \xi$  term: Recall the definitions for  $B^0$  (3.7),  $B^1$  (3.8), and  $E$  (4.17). Using Cauchy-Schwarz, corollary 4.3.2, and the boundedness of  $\frac{n^2}{m^2} B^\alpha$  on  $(0, y_*^0) \times (-y_*^1, y_*^1)$  we have

$$\left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} B^\alpha \partial_\alpha \xi \cdot \partial_{y^1} \tilde{F}_0 \right| \lesssim E^{1/2} \|\partial_{y^1} \tilde{F}_0\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\sqrt{\epsilon}} E^{1/2} \underline{A}$$

(b)  $L_\epsilon(\tilde{F}_0, R) \xi$  term: Integrating by parts with respect to  $y^1$  twice, using that  $\text{Hess}_\Phi W(\tilde{F}_0, R)$  is symmetric, using that  $\partial_{y^1} \tilde{F}_0 \in \ker(L_\epsilon(\tilde{F}_0, R))$ , and using that on  $(0, y_*^0) \times (-y_*^1, y_*^1)$ ,  $\frac{n^2}{m^2}$  is bounded we have

$$\begin{aligned} \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} L_\epsilon(\tilde{F}_0, R) \xi \cdot \partial_{y^1} \tilde{F}_0 \right| &\leq \left| \int_{|y^1| \leq y_*^1} \partial_{y^1 y^1} \left( \frac{n^2}{m^2} \right) \xi \cdot \partial_{y^1} \tilde{F}_0 \right| + \left| \int_{|y^1| \leq y_*^1} 2 \partial_{y^1} \left( \frac{n^2}{m^2} \right) \xi \cdot \partial_{y^1 y^1} \tilde{F}_0 \right| \\ &+ \left| \left[ \partial_{y^1} \xi \cdot \partial_{y^1} \tilde{F}_0 + \xi \cdot \partial_{y^1} \left( \frac{n^2}{m^2} \right) \partial_{y^1} \tilde{F}_0 \right]_{-y_*^1}^{y_*^1} \right| \\ &\leq \frac{1}{\sqrt{\epsilon}} E^{1/2} + \frac{1}{\epsilon^2} e^{-\alpha \frac{y_*^1}{\epsilon}} e^{\alpha \underline{A}} \end{aligned}$$

(c)  $S_{-1}$  term: Using the definition of  $S_{-1}$  (4.12), Cauchy-Schwarz, corollary 4.3.2, using the fact that  $B^1 = -H(R) + O(y^1)$ , and the boundedness of  $\frac{n^2}{m^2}$  on  $(0, y_*^0) \times (-y_*^1, y_*^1)$  we have

$$\begin{aligned} \left| \int_{|y^1| \leq y_*^1} S_{-1} \cdot \partial_{y^1} \tilde{F}_0 \right| &= \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} \partial_{y^1} \tilde{F}_0 \cdot [B^1 + H(R)] \partial_{y^1} \tilde{F}_0 + \frac{n^2}{m} \frac{a}{\epsilon} \partial_R W(\tilde{F}_0, R) \cdot \partial_{y^1} \tilde{F}_0 \right| \\ &\lesssim \int_{-y_*^1}^{y_*^1} |y^1| |\partial_{y^1} \tilde{F}_0|^2 + \frac{|a|}{\epsilon} \|\partial_R W(\tilde{F}_0, R)\|_{L^2(\mathbb{R})} \|\partial_{y^1} \tilde{F}_0\|_{L^2(\mathbb{R})} \\ &\lesssim \underline{A} \end{aligned}$$

- (d)  $N$  term: To estimate this term we proceed as we did in the energy estimate when we estimated the  $N \cdot \partial_{y^0} \xi$  term in lemma 4.3.6, where  $N$  was defined in (4.14). To obtain this estimate, we again use the identity  $g(t) = g(0) + g'(0)t + \int_0^1 (1-t)g''(t)$  to rewrite  $N$  as

$$N = -\frac{1}{\epsilon^2} \int_0^1 (1-t) \frac{d^2}{dt^2} w(\tilde{F}_0 + \tilde{F}_\xi, R + ty^1 m) dt$$

Thus,

$$\begin{aligned} \epsilon^2 \left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} N \cdot \partial_{y^1} \tilde{F}_0 \right| &= \epsilon^2 \left| \left\langle \frac{n^2}{m^2} N, \partial_{y^1} \tilde{F}_0 \right\rangle_{L^2(-y_*^1, y_*^1)} \right| \\ &\leq \max_{t \in [0,1]} \left| \left\langle \frac{n^2}{m^2} \frac{d^2}{dt^2} w(\tilde{F}_0 + \tilde{F}_\xi, R + ty^1 m), \partial_{y^1} \tilde{F}_0 \right\rangle_{L^2(-y_*^1, y_*^1)} \right| \\ &\leq \max_{t \in [0,1]} \left| \left\langle \left( (\tilde{F}_\xi)_\phi \partial_\phi + (\tilde{F}_\xi)_\sigma \partial_\sigma \right) \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi, \partial_{y^1} \tilde{F}_0 \right\rangle_{L^2(-y_*^1, y_*^1)} \right| \\ &\quad + \max_{t \in [0,1]} \left| \left\langle 2y^1 m(y^0) \partial_R \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi, \partial_{y^1} \tilde{F}_0 \right\rangle_{L^2(-y_*^1, y_*^1)} \right| \\ &\quad + \max_{t \in [0,1]} \left| \left\langle (y^1 m)^2 \partial_{RR} w, \partial_{y^1} \tilde{F}_0 \right\rangle_{L^2(-y_*^1, y_*^1)} \right| \end{aligned}$$

and hence

$$\begin{aligned} &\left| \int_{|y^1| \leq y_*^1} \frac{n^2}{m^2} N \cdot \partial_{y^1} \tilde{F}_0 \right| \\ &\lesssim \frac{1}{\epsilon^2} \|\tilde{F}_\xi\|_\infty \|\tilde{F}_\xi\|_{L^2(-y_*^1, y_*^1)} \|\partial_{y^1} \tilde{F}_0\|_{L^2(\mathbb{R})} + \frac{1}{\epsilon^2} \|\tilde{F}_\xi\|_{L^2(-y_*^1, y_*^1)} \|y^1 \partial_{y^1} \tilde{F}_0\|_{L^2(\mathbb{R})} + \frac{1}{\epsilon^2} \int_{-y_*^1}^{y_*^1} (y^1)^2 \tilde{s}_0 \partial_{y^1} \tilde{s}_0 \\ &\lesssim \frac{1}{\epsilon^{5/2}} (\epsilon + \|\xi\|_\infty) (\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) + \frac{1}{\epsilon^{3/2}} (\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}}) + 1 \end{aligned}$$

We are left to estimate  $\|\xi\|_\infty$ . Using Gagliardo-Nirenberg we get that

$$\begin{aligned} \|\xi\|_\infty &\lesssim \|\xi\|_{L^2(-y_*^1, y_*^1)}^{1/2} \|\partial_{y^1} \xi\|_{L^2(-y_*^1, y_*^1)}^{1/2} \\ &\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}})^{1/2} E^{1/4} \end{aligned}$$

Thus, we have that

$$\left| \int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2} N \cdot \partial_{y^1} \tilde{F}_0 \right| \lesssim 1 + \frac{1}{\epsilon^{5/2}} (\epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}})^{1/2} E^{1/4}) (\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{y_*^1 - a}{\epsilon}})$$

4. Finally, recall the definition of  $E$  (4.17). Using (4.52), Cauchy-Schwarz, and corollary 4.3.2 we have

$$\left| \int_{|y^1| \leq y_*^1} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_1 \right| \lesssim \frac{1}{\sqrt{\epsilon}} \left( E + \frac{1}{\epsilon} e^{-\alpha \frac{y_*^1 - a}{\epsilon}} \right) \left[ 1 + \frac{|a''|}{\epsilon} \right] \underline{A}$$

Combining the estimates obtained in steps 1-4 we get that

$$\frac{|a''|}{\epsilon} \lesssim \frac{|a''|}{\epsilon} \left[ \epsilon \underline{A} + \sqrt{\epsilon} \underline{A} \sqrt{E} + e^{-\alpha \frac{y_*^{1-a}}{\epsilon}} \right] + \left[ \frac{1}{\sqrt{\epsilon^5}} (\epsilon \underline{A} + (\epsilon \sqrt{E} + e^{-\alpha \frac{y_*^{1-a}}{\epsilon}})^{1/2} \sqrt{E}) (\sqrt{\epsilon^3} + \epsilon \sqrt{E} + e^{-\alpha \frac{y_*^{1-a}}{\epsilon}}) \right]$$

as desired. This concludes the proof of theorem 4.2.3.  $\square$

## A Formal Asymptotics

Let  $\eta$  be the Minkowski metric on  $\mathbb{R}^{1+n}$  and let  $\Gamma \subset (\mathbb{R}^{1+n}, \eta)$  be an  $n$ -dimensional time-like surface in space-time. Suppose that  $\Gamma$  is parameterized by some map  $H : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^{1+n}$ . Define a new coordinate system  $(y^\tau, y^\nu) \in \mathbb{R}^n \times \mathbb{R}$ , called Minkowski normal coordinates, as

$$(t, x) = \psi(y^\tau, y^\nu) = H(y^\tau) + y^\nu \nu(y^\tau)$$

where  $\nu(y^\tau) \perp_{\eta} \partial_{y^\tau} H(y^\tau)$  and  $|\nu(y^\tau)|_{\eta} = 1$ . We call  $y^\tau \in \mathbb{R}^n$  “tangential coordinates” and  $y^\nu \in \mathbb{R}$  the “normal coordinate”. Note that this coordinate system may only be well defined on a neighbourhood  $\mathcal{N}$  of  $\Gamma$ .

Recall that we want to find solutions of (1.1) so that  $\phi$  has an interface and so that  $\sigma$  is exponentially small except near the interface of  $\phi$ . Based on [14], we expect that for suitable  $\Gamma$ ,  $\theta : \Omega \rightarrow \mathbb{R}$ , and  $\Phi_0 := (\phi_0, \sigma_0) : \mathbb{R} \rightarrow \mathbb{R}^2$  there exists a solution with these characteristics of the form

$$\Phi(y^\tau, y^\nu) \approx \begin{pmatrix} \phi_0(\frac{y^\nu}{\epsilon}) \\ e^{i\theta(y^\tau)} \sigma_0(\frac{y^\nu}{\epsilon}) \end{pmatrix} \quad (\text{A.1})$$

where  $\gamma_{ij} := \eta_{\alpha\beta} \partial_i H^\alpha \partial_j H^\beta$  is the induced metric on the surface  $\Gamma$  (latin indices range over the tangential coordinates and Greek indices will range over both tangential and normal coordinates).

We will now carry out a formal asymptotic analysis to find  $\Phi_0$  so that  $\phi_0$  has an interface and to find  $\Gamma$  and  $\theta$  for which we expect (A.1) to hold. To do this, we will expand the action integral associated to (1.1) about the right hand side of (A.1). From this expansion, we obtain an **effective action**. We will then make a choice for the profile  $\Phi_0$  and for this choice of  $\Phi_0$ , we expect, heuristically, that the correction terms coming from expanding the action about the right hand side of (A.1) will be of lower order when  $\Gamma$  and  $\theta$  are critical points of the effective action.

The Lagrangian associated to (1.1) in Minkowski normal coordinates is

$$\mathcal{L} := \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{2} g^{\alpha\beta} \overline{\partial_\alpha \sigma} \partial_\beta \sigma + \frac{1}{\epsilon^2} V(\phi, \sigma) \quad (\text{A.2})$$

where  $g_{\alpha\beta} := \eta_{\lambda\omega} \partial_\alpha \psi^\lambda \partial_\beta \psi^\omega$  is the Minkowski metric in normal coordinates. Note that

$$g_{\alpha\beta} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & 1 \end{pmatrix} + (y^\nu)^2 \begin{pmatrix} \eta_{\lambda\omega} \partial_i \nu^\lambda \partial_j \nu^\omega & 0 \\ 0 & 0 \end{pmatrix}$$

For  $\xi = (\xi_\phi, \xi_\sigma) : \mathbb{R}^{1+n} \rightarrow \mathbb{R} \times \mathbb{C}$ , we plug

$$\phi = \phi_0\left(\frac{y^\nu}{\epsilon}\right) + \xi \quad \text{and} \quad \sigma = e^{i\theta(y^\tau)} \left[ \sigma_0\left(\frac{y^\nu}{\epsilon}\right) + \xi_\sigma \right]$$

into the action integral to get

$$S(\Phi) = \frac{1}{\epsilon^2} \int \left\{ \frac{1}{2} (\Phi'_0(\frac{y^\nu}{\epsilon}))^2 + V(\Phi_0)(\frac{y^\nu}{\epsilon}) + \frac{1}{2} \gamma^{ij} \partial_i \theta \partial_j \theta \sigma_0(\frac{y^\nu}{\epsilon})^2 \right\} \sqrt{-\gamma(y^\tau)} dy^\tau dy^\nu + \text{other terms} \quad (\text{A.3})$$

The effective action we obtain from this expansion is

$$\tilde{S} := \int \left\{ \frac{1}{2} (\Phi'_0(\frac{y^\nu}{\epsilon}))^2 + V(\Phi_0)(\frac{y^\nu}{\epsilon}) + \frac{1}{2} \gamma^{ij} \partial_i \theta \partial_j \theta \sigma_0(\frac{y^\nu}{\epsilon})^2 \right\} \sqrt{-\gamma(y^\tau)} dy^\tau dy^\nu \quad (\text{A.4})$$

Consider the  $\frac{1}{\epsilon^2}$  term. It is natural to choose  $\Phi_0$  so that in transverse directions to  $\Gamma$ ,  $\Phi_0$  is energy minimizing and so that  $\phi_0$  has an interface. To this end, suppose for  $\rho \in \mathbb{R}$ ,  $F = (f, s)(\cdot; \rho)$  satisfies the minimization problem

$$\mu(\rho) := \inf_{(f,s) \in \mathcal{A}} \int \left\{ \frac{1}{2} |(f', s')|^2 + V(f, s) + \frac{1}{2} \rho s^2 \right\} dy^\nu \quad (\text{A.5})$$

$$\mathcal{A} := \left\{ (f, s) \in C^1 : \lim_{y^\nu \rightarrow \pm\infty} f(y^\nu) = \pm 1 \right\} \quad (\text{A.6})$$

In this case, the boundary conditions imposed results in  $f$  having an interface. Furthermore, for suitable potentials  $V$ ,  $s$  is exponentially small except near the interface of  $f$ . We pick  $\Phi_0 = (f, s)(\cdot; \zeta)$ , where  $\zeta(y^\tau) := \gamma^{ij} \partial_i \theta \partial_j \theta$ . *Important:* The natural choice of profile  $\Phi_0$  actually depends on  $\zeta$ . That is, in contrast to our initial hypothesis (A.1), we expect that there should exist a solution to (??) satisfying

$$\Phi \approx \begin{pmatrix} \phi_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \\ e^{i\theta(y^\tau)} \sigma_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \end{pmatrix} \quad (\text{A.7})$$

for  $\Phi_0 = \Phi_0(\cdot; \zeta)$  minimizing (A.5) and for suitable  $\Gamma$  and  $\theta$ .

For this choice of  $\Phi_0$ , the effective action becomes

$$\tilde{S}(H, \theta) = \int \mu(\zeta) \sqrt{-\gamma} dy^\tau \quad (\text{A.8})$$

Heuristically, we expect that when  $\theta$  and  $H$  are critical points of  $\tilde{S}$ , then  $\xi$  will be of lower order than the right hand side of (A.1). That is, for  $\theta$  and  $H$  satisfying the nonlinear, coupled system

$$0 = \frac{\delta \tilde{S}}{\delta \theta} = -2 \partial_j \left( \mu'(\zeta) \sqrt{-\gamma} \gamma^{ij} \partial_i \theta \right) \quad (\text{A.9})$$

$$0 = \frac{\delta S}{\delta H} = -\eta_{\alpha\beta} \partial_j \left( \mu(\zeta) \sqrt{-\gamma} \gamma^{ij} \partial_i H^\alpha \right) + 2 \eta_{\alpha\beta} \partial_j \left( \mu'(\zeta) \sqrt{-\gamma} \gamma^{ik} \gamma^{lj} \partial_k \theta \partial_l \theta \partial_i H^\beta \right) \quad (\text{A.10})$$

then  $\Phi_0(\frac{y^\nu}{\epsilon}; \zeta)$  should be a good approximate solution. The coupled system for  $\theta$  and  $H$  should be a hyperbolic system, but this isn't completely clear. By expanding (A.10) and taking its inner product with  $\nu^\beta$ , we can rewrite this system as

$$\square_\Gamma \theta = -\gamma(\nabla_\tau \log [\mu'(\zeta)], \nabla_\tau \theta) \quad (\text{A.11})$$

$$\text{mean curvature of } \Gamma = 2 \frac{\mu'(\zeta)}{\mu(\zeta)} \mathbb{I}(\nabla_\tau \theta, \nabla_\tau \theta) \quad (\text{A.12})$$

where  $\mathbb{I}$  is the second fundamental form of  $\Gamma$ . From (A.12) we find a nice geometric relation between the surface about which our approximate solution is concentrated and the phase of  $\sigma_0$ .

## 5 References

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